

Preface

The overriding objective of this book is to offer a review of vector calculus needed for the physical sciences and engineering. This review includes necessary excursions into tensor analysis intended as the reader's first exposure to tensors, making aspects of them understandable at the undergraduate level. A secondary objective of this book is to prepare the reader for more advanced studies in these areas.

As the world embarks on new horizons in photonics and materials science, honing one's skills in vector calculus and learning the essential role that tensors play are paramount. New inroads in engineering are driving the need for a revamp of engineering mathematics in these areas. Profound new paradigms in optical engineering and new advances in composites are necessitating these changes. The author has found that there is an ever-increasing need for vector calculus concepts to be extended to tensors and that his undergraduates can indeed grasp tensorial concepts if taught following the lines of thinking presented here.

Whereas the classical approach to teaching electromagnetics at the junior level has been to avoid any mention of tensors, the high-tech world entering the third millennium warrants a rethinking of this practice. This is especially true as nonlinear optical effects become more common in the design of optical systems. Advanced materials, especially composites and nanodesigned materials, provide further evidence supporting the teaching of tensor fundamentals to upper-division* students. Even for isotropic materials, the fundamental relationship between stress, strain, and elastic modulus—which are rank-two and rank-four tensors—requires a fundamental understanding of tensor analysis. For anisotropic materials such as composites, piezoelectric materials, and magnetostrictive materials, tensorial relationships are unavoidable even in the linear regime.

Furthermore, the development of new photonics devices in optoelectronics, acousto-optics, magneto-optics, and fiber optics is playing an ever-increasing role in contemporary communications system design.^{1,2,3,4} Pollock states

* University-level juniors and seniors.

A surface in 3D space given in generalized orthogonal curvilinear coordinates q_1, q_2, q_3 is described as $h_3 q_3 = f(h_1 dq_1, h_2 dq_2)$. Then the change in f with respect to $d\ell_1 = h_1 dq_1$, for example, is

$$\frac{\partial f}{d\ell_1} = \frac{\partial f}{h_1 \partial q_1} = \frac{\partial(h_3 q_3)}{h_1 \partial q_1} = \frac{h_3}{h_1} \frac{\partial q_3}{\partial q_1} + \frac{q_3}{h_1} \frac{\partial h_3}{\partial q_1} \quad (1.3-17)$$

where the metric coefficients are in general functions of all three coordinates $h_i = h_i(q_1, q_2, q_3)$ and therefore must be included in the derivative when applying the product rule.

1.3.4 Partial derivative of a vector function

By the time students in the physical sciences or engineering enter upper-division courses (junior and senior years of a bachelor's program), they will have been exposed to the partial derivative. However, this introduction was invariably done in Cartesian coordinates with Cartesian-coordinate examples. This was fine when the partial derivative being explained was taken on a scalar function. However, problems can arise if the partial derivative is taken of a vector function (or of any tensor of rank greater than zero) and the physical problem leads naturally into curvilinear coordinates, such as cylindrical coordinates. We will therefore discuss partial derivatives of vectors in generalized orthogonal curvilinear coordinates.

Let us examine the partial derivative of a vector field $\bar{A}(q_1, q_2, q_3) = \hat{u}_1 A_1(q_1, q_2, q_3) + \hat{u}_2 A_2(q_1, q_2, q_3) + \hat{u}_3 A_3(q_1, q_2, q_3)$. The partial derivative of \bar{A} with respect to one of the coordinates q_i is

$$\begin{aligned} \frac{\partial \bar{A}}{\partial q_i} &= \frac{\partial(\hat{u}_1 A_1)}{\partial q_i} + \frac{\partial(\hat{u}_2 A_2)}{\partial q_i} + \frac{\partial(\hat{u}_3 A_3)}{\partial q_i} \\ &= \hat{u}_1 \frac{\partial A_1}{\partial q_i} + \hat{u}_2 \frac{\partial A_2}{\partial q_i} + \hat{u}_3 \frac{\partial A_3}{\partial q_i} \\ &\quad + A_1 \frac{\partial \hat{u}_1}{\partial q_i} + A_2 \frac{\partial \hat{u}_2}{\partial q_i} + A_3 \frac{\partial \hat{u}_3}{\partial q_i} \end{aligned} \quad (1.3-18)$$

where the first three terms on the right-hand side involve partial derivatives of the scalar components $A_1(q_1, q_2, q_3)$, $A_2(q_1, q_2, q_3)$, and $A_3(q_1, q_2, q_3)$ of the vector field, each in their respective unit-vector direction. These terms are therefore handled as in Eqs. (1.3-1), (1.3-3), and (1.3-5). The last three terms

involve coordinate derivatives of unit vectors and must be considered—a point entirely missed when the Cartesian system is used.

The vital difference is that spatial derivatives of unit vectors in Cartesian coordinates are all zero, but that derivatives of unit vectors with respect to coordinates that are curved in space often are not. One might think that the derivative of a vector whose length is constant has to be zero. However, this is not the case.

In general it can be shown¹¹ that

$$\frac{\partial \hat{u}_i}{\partial q_i} = -\frac{\hat{u}_j}{h_j} \frac{\partial h_i}{\partial q_j} - \frac{\hat{u}_k}{h_k} \frac{\partial h_i}{\partial q_k} \quad (1.3-19)$$

and

$$\frac{\partial \hat{u}_i}{\partial q_j} = \frac{\hat{u}_j}{h_i} \frac{\partial h_j}{\partial q_i} \quad (1.3-20)$$

where $i = 1, 2, 3$; $j = 2, 3, 1$ and $k = 3, 1, 2$, in that order. Further, if the derivative of a unit vector is not zero, it will always be at right angles to that unit vector. Thus,

$$\hat{u}_i \cdot \frac{\partial \hat{u}_i}{\partial q_j} = 0 \quad (1.3-21)$$

Example: The movement of a clock hand to illustrate the need for coordinate derivative of a unit vector.

Think of the hand of a clock. In a cylindrical coordinate system (or just a polar coordinate system because the problem is just 2D), our coordinates q_1 and q_2 are r and ϕ , and the metric coefficients h_1 and h_2 are 1 and r , respectively. Representing the clock hand as \hat{u}_r , the ϕ -coordinate partial derivative of \hat{u}_r can be found from (1.3-20):

$$\begin{aligned} \frac{\partial \hat{u}_r}{\partial q_\phi} &= \frac{\hat{u}_\phi}{h_r} \frac{\partial h_\phi}{\partial r} = \frac{\hat{u}_\phi}{1} \frac{\partial r}{\partial r} \\ &= \hat{u}_\phi \end{aligned} \quad (1.3-22)$$

Therefore, the rate of change of the clock hand, represented by the unit vector in the r -direction with respect to the azimuthal ϕ direction is equal to the unit vector in the ϕ direction. Further, by applying this result to (1.3-21), we see that

$$\hat{u}_r \cdot \frac{\partial \hat{u}_r}{\partial \phi} = \hat{u}_r \cdot \hat{u}_\phi = 0 \quad (1.3-23)$$

Thus, the derivative of the unit vector that is always pointing in the direction of the clock hand, i.e. \hat{u}_r , with respect to ϕ is at right angles to \hat{u}_r and, in fact, is in the \hat{u}_ϕ direction. This orthogonal result will always be so because the unit vector being differentiated does not change length.

References

1. Douglas H. Werner and Raj Mittra, *Frontiers in Electromagnetics*, IEEE Press, Piscataway, NJ (2000).
2. Max Born and Emil Wolf, *Principles of Optics*, 6th ed., Cambridge University Press (1980).
3. Govind P. Agrawal, *Nonlinear Fiber Optics*, 3rd Ed., Academic Press, London (2001), pp. 17ff.
4. Clifford R. Pollock, *Fundamentals of Optoelectronics*, McGraw-Hill, New York (1994), pp. 190,200, e.g.
5. Robert W. Boyd, *Nonlinear Optics*, Academic Press, London (1991).
6. Paul N. Butcher and David Cotter, *The Elements of Nonlinear Optics*, Cambridge University Press (1991), p. 4ff.
7. Eugenio Iannone, Francesco Matera, Antonio Mecozzi and Marina Settembre, *Nonlinear Optical Communication Networks*, Wiley, New York (1998), p. 36ff.
8. William H. Hayt, Jr. and John A. Buck, *Engineering Electromagnetics*, 6th ed., McGraw-Hill, New York, (2001).
9. Matthew N. O. Sadiku, *Elements of Electromagnetics*, 3rd ed., Oxford University Press, (2000).
10. Clayton R. Paul, *Electromagnetics for Engineers: With Applications to Digital Systems and Electromagnetic Interference*, Wiley, New York (2003).
11. A. I. Lur'e, *Three Dimensional Problems of the Theory of Elasticity*, Wiley Interscience, Hoboken, NJ (1964).

$$\int_A \vec{F} \cdot \vec{da} \quad (2.4-23)$$

where the area A is bounded (by a closed line).

When the surface integral is closed to enclose a volume, Eq. (2.4-23) takes the form

$$\oint \vec{F} \cdot \vec{da} \quad (2.4-24)$$

These integrals with dot products in the integrand are frequently used in disciplines of mathematical physics, such as quantum physics and electromagnetics. The dot product in the integrand is simply a convenient way to sum only the component of \vec{F} at each differential element of surface over which the integration takes place that lies normal to that surface element.

Examples of Eqs. (2.4-23) and (2.4-24) can be found in Sections 5.2.2, 5.3.1, and 5.3.4.

2.4.3(c) Cross product and the Levi-Civita symbol

The “cross” product of vector \vec{A} with another vector \vec{B} is spoken as “ \vec{A} cross \vec{B} ” and written as $\vec{A} \times \vec{B}$. The cross product is defined by

$$\vec{A} \times \vec{B} = \hat{u}_{\perp_{A \rightarrow B}} |A| |B| \sin \theta_{AB} \quad (2.4-25)$$

where $\hat{u}_{\perp_{A \rightarrow B}}$ is a unit vector normal to the plane containing \vec{B} and \vec{A} and is in a direction given in a right-hand sense—namely by aligning the fingers of your right hand along the direction of \vec{A} and turning them into the direction of \vec{B} so that your thumb points in the direction of $\hat{u}_{\perp_{A \rightarrow B}}$. The angle θ_{AB} is the angle made in so doing.

(i) Commutative and distributive laws for cross products

From Eq. (2.4-25), note that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. Thus, the commutative law does not hold for the cross product operation. However, the *distributive law* does hold. Namely,

$$\begin{aligned}
& (\bar{A} + \bar{B} + \dots) \times (\bar{M} + \bar{N} + \dots) \\
&= \bar{A} \times \bar{M} + \bar{A} \times \bar{N} + \dots \\
&+ \bar{B} \times \bar{M} + \bar{B} \times \bar{N} + \dots \\
&+ \dots
\end{aligned} \tag{2.4-26}$$

(ii) Vector cross products and the Levi-Civita symbol

Unit vectors in each of three orthogonal directions $\hat{u}_1, \hat{u}_2, \hat{u}_3$ have well-determined cross-product relationships. These relationships are described conventionally in the following paragraph and described with the elegance of the *Levi-Civita* symbol in the subsequent paragraph.

The cross product of unit vectors in 3D space becomes trivalued, namely, $-1, 0,$ and $+1$. The usual process used in sophomore-level texts to explain this trivalued system is to first point out that $\hat{u}_i \times \hat{u}_i = 0$ because $\theta_{ij} = 0$ and the $\sin \theta_{ij} = 0$ in Eq. (2.4-25). Further, $\hat{u}_i \times \hat{u}_{i+1} = +\hat{u}_{i+2}$ where $i = 1, 2, 3; i + 1 = 2, 3, 1$ and $i + 2 = 3, 1, 2$, because $\theta_{(i)(i+1)} = \pi/2$ and $\sin \theta_{(i)(i+1)} = 1$. The right-hand rule specifies that direction 1 crossed into direction 2 yields positive direction 3, or direction 2 crossed into direction 3 yields positive direction 1, and direction 3 crossed into direction 1 yields positive direction 2. However, $\hat{u}_i \times \hat{u}_{i+2} = -\hat{u}_{i+1}$, where $i = 1, 2, 3; i + 2 = 3, 1, 2;$ and $i + 1 = 2, 3, 1$. The minus sign is needed because when direction 1 is crossed into direction 3 the thumb points opposite to (or the negative of) direction 2. Likewise, 2 into 1 yields the negative of direction 3 and 3 into 2 yields the negative of direction 1. The angle from 1 to 3 may be taken as $-\pi/2$ since the angle from 3 to 1 is $\pi/2$. Thus, $\theta_{(i)(i+2)} = -\pi/2$ and $\sin \theta_{(i)(i+2)} = -1$.

However, the *Levi-Civita* symbol ϵ_{ijk} shortcuts the discussion in the preceding paragraph. If one calls the sequence 1,2,3,1,2 *cyclic*, the sequence 3,2,1,3,2 *acyclic*, and cases where any two adjacent indices are the same *noncyclic*, we define the Levi-Civita symbol as¹⁴

$$\epsilon_{ijk} = \begin{cases} 1 & \text{cyclic} \\ 0 & \text{noncyclic} \\ -1 & \text{acyclic} \end{cases} \tag{2.4-27}$$

and therefore,

$$\mathbf{u}_i \times \mathbf{u}_j = \epsilon_{ijk} \mathbf{u}_k \tag{2.4-28}$$

which is a tensor notation formulation with the unit vector hats implied.

The cross product of our vector \bar{A} with \bar{B} in tensor notation can then be defined as

$$A_i u_i \times B_j u_j = A_i B_j \epsilon_{ijk} u_k \quad (2.4-29)$$

(iii) Area formulas using cross products

In Section 1.2 differential area was defined and discussed without the benefit of the cross product. A description of the vector differential area [Eq. (1.2-5)] can now be expressed as

$$\bar{da} = \hat{u}_k \left| \bar{d\ell}_i \right| \left| \bar{d\ell}_j \right| = \frac{\bar{d\ell}_i \times \bar{d\ell}_j}{\left| \bar{d\ell}_i \times \bar{d\ell}_j \right|} \left| \bar{d\ell}_i \right| \left| \bar{d\ell}_j \right| \quad (2.4-30)$$

or more simply in tensor notation as

$$\bar{da} = d\ell_i u_i \times d\ell_j u_j = d\ell_i d\ell_j \epsilon_{ijk} u_k \quad (2.4-31)$$

Note also that the area of the parallelogram with adjacent sides \bar{A} and \bar{B} is the magnitude of the cross product where

$$\text{Area} = |A| |B| \sin \theta_{AB} = |\bar{A} \times \bar{B}| \quad (2.4-32)$$

This is illustrated in Fig. 2.4-2.

Other applications of the cross product include finding the moment of a force acting at a distance, finding the force on a current-carrying conductor in a magnetic field, and dealing with the mechanics of gyroscopes, among many others.

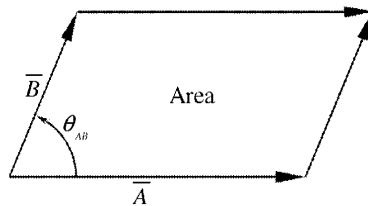


Figure 2.4-2 The area of a parallelogram as $|\bar{A} \times \bar{B}|$.

(iv) Cross product coordinate expansion

Using the same vectors $\bar{A} = \hat{u}_i A_i$ and $\bar{B} = \hat{u}_j B_j$ as before, but using tensor notation, the cross product takes the form

$$\begin{aligned}
 \bar{A} \times \bar{B} = & \overset{0}{\cancel{\hat{u}_1 \times \hat{u}_1}} A_1 B_1 + \overset{\hat{u}_3}{\hat{u}_1 \times \hat{u}_2} A_1 B_2 + \overset{-\hat{u}_2}{\hat{u}_1 \times \hat{u}_3} A_1 B_3 \\
 & + \overset{-\hat{u}_3}{\hat{u}_2 \times \hat{u}_1} A_2 B_1 + \overset{0}{\cancel{\hat{u}_2 \times \hat{u}_2}} A_2 B_2 + \overset{\hat{u}_1}{\hat{u}_2 \times \hat{u}_3} A_2 B_3 \\
 & + \overset{\hat{u}_2}{\hat{u}_3 \times \hat{u}_1} A_3 B_1 + \overset{-\hat{u}_1}{\hat{u}_3 \times \hat{u}_2} A_3 B_2 + \overset{0}{\cancel{\hat{u}_3 \times \hat{u}_3}} A_3 B_3
 \end{aligned} \tag{2.4-33}$$

From Eq. (2.4-28), the cross product factors become 0, \hat{u}_k , or $-\hat{u}_k$, where $k = 1, 2, 3$, as shown in Eq. (2.4-33).

Collecting terms in each of the three coordinate directions,

$$\begin{aligned}
 \bar{A} \times \bar{B} = & \hat{u}_1 (A_2 B_3 - A_3 B_2) \\
 & + \hat{u}_2 (A_3 B_1 - A_1 B_3) \\
 & + \hat{u}_3 (A_1 B_2 - A_2 B_1)
 \end{aligned} \tag{2.4-34}$$

Notice that this can also be represented in determinate form as

$$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{u}_1 & \hat{u}_2 & \hat{u}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{2.4-35}$$

Alternatively, tensor notation can be used in conjunction with the Levi-Civita symbol to express $\bar{A} \times \bar{B}$ as

$$(\hat{u}_i A_i) \times (\hat{u}_j B_j) = \hat{u}_k \epsilon_{ijk} A_i B_j \tag{2.4-36}$$

in its ultimate beauty and simplicity, but still preserving all of the operations of Eq. (2.4-33) resulting in the six nonzero terms of (2.4-34), including the three minus signs.

$$\lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \left[\int_{\text{front}} + \int_{\text{back}} \right] = \frac{1}{h_1 h_2 h_3} \frac{\partial(h_2 h_3 A_1)}{\partial q_1} \quad (4.4-21)$$

for one of the three scalar terms making up the divergence of the vector field \bar{A} .

The same process may be repeated for the other two pairs of integrals; however, this tedious procedure is not necessary because we may simply *roll the subscripts** to obtain the remaining two terms of the divergence. The *roll* sequence is $1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2$. Therefore, the divergence of our vector field \bar{A} is

$$\begin{aligned} \nabla \cdot \bar{A} &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(h_2 h_3 A_1)}{\partial q_1} + \frac{\partial(h_3 h_1 A_2)}{\partial q_2} + \frac{\partial(h_1 h_2 A_3)}{\partial q_3} \right] \\ &= \frac{1}{h_1 h_2 h_3} \sum_{i=1}^3 \frac{\partial}{\partial q_i} \left(\frac{h_1 h_2 h_3}{h_i} A_i \right) \end{aligned} \quad (4.4-22)$$

This equation is specialized for Cartesian coordinates in Appendix B, Eq. (B.1-3) and for cylindrical coordinates in Eq. (B.3-9).

4.5 The Curl Differential Operator

The *curl operator* is the third of the three first-order vector differential operators introduced in Section 4.1. Whereas the gradient employed the del operator (∇) directly and the divergence employed the del-dot operator ($\nabla \cdot$), the curl employs the *del-cross operator*, denoted by “ $\nabla \times$ ”. In the previous section, we found that divergence of a vector could not in general be found by simply taking the dot product of the del operator with the vector because it was necessary to account for variations in surface elements as well as the vector components. Here we will find a similar admonition. The curl of a vector field is not simply the cross product of the del operator with the vector for a similar reason. Although one can validly get by with this misleading approach when expanding the curl in

* One of the paramount advantages of using generalized coordinates (GOCCs) is the ability to roll subscripts. When expanding vector operators into GOCCs in 3D space, it is necessary to do so for one third of the problem. The remaining two-thirds may be deduced by simply rolling the subscripts. This procedure is invalid in all specific (nongeneralized) coordinate systems except Cartesian coordinates. In Cartesian coordinates, it is permissible because all three metric coefficients, h_x, h_y, h_z , are unity and thus do not have variations with respect to coordinate directions. In this special case, the roll sequence is $x \rightarrow y \rightarrow z \rightarrow x \rightarrow y$.

Cartesian coordinates, it is invalid to do so in any other coordinate system. Many texts that first introduce the student to vector differential operators immediately fall into this oversimplistic approach of expanding these operators in rectangular coordinates, no doubt because to immediately expand into GOCCs exposes the student to an inordinate complexity of calculative rules before providing a perception of the nature of the operator. In the case of the curl, Bevc states this with his usual insight and precision:¹²

To be sure, such rules are useful in actual calculations but they hardly provide any physical insight into the nature of the curl and moreover depend on coordinate systems.

In this section we find that the curl does not change the rank of the field upon which it operates. This is in contrast to the previous two operators in that the result of a gradient operation added one to the rank and the divergence subtracted one from the rank of field that was operated upon. Therefore, if the three operators act on a vector field, the divergence will yield a scalar, the curl will yield a vector, and the gradient will yield a dyadic.

Like the gradient and the divergence, the curl is a first-order vector operator using the del notation; however, the similarities end there. The curl operator is entirely different from the two just previously reviewed. The inherent definitions are based on three entirely different geometries—the gradient on a differential length going to zero in the limit, the curl on a differential area going to zero in the limit (as we will soon see), and the divergence on a differential volume going to zero in the limit. In addition, *the curl operates transversely, whereas the divergence operates tangentially and the gradient operates omniversely, so-to-speak*. By this, we do not mean that the resultant is transverse, tangential, etc.—just the operand acts in these ways.

4.5.1 The curl of a vector field—a physical description

Let us next acquire a physical understanding of this vector field that we call the *curl of a vector field* from the definition (outside of the context of coordinates). Again, the definition of this vector operation is given by Bevc¹² (with emphasis added):

The curl of a vector field \vec{A} at a point is a vector pointing in the direction of a normal to an infinitesimal surface which is so oriented in space that the limit of the ratio of the line integral of the vector

field \bar{A} around the perimeter of that surface to the area enclosed is **maximal**. The magnitude of the curl is the value of that limit.

Mathematically, the curl vector \bar{C} of the vector field \bar{A} is by this definition determined by

$$\bar{C} = \text{curl } \bar{A} \equiv \lim_{\Delta a \rightarrow 0} \left. \frac{\oint \bar{A} \cdot d\bar{\ell}}{\Delta a} \right|_{\max} \hat{u}_n \quad (4.5-1)$$

where

$$\lim_{\Delta a \rightarrow 0} (\text{circ}(\bar{A})/\Delta a) \Big|_{\max}$$

is the maximum of the ratio of the *circulation* of $\bar{A}(\bar{r}, t)$, defined by Eq. (2.4-22), about the point P located at \bar{r} in space and at time t to the enclosed area, and where \hat{u}_n is the normal to that surface at P in the right-hand sense with respect to the direction of the closed-line integration. Shorthand notation for the curl is given by the use of the *del-cross operator* as

$$\bar{C} = \nabla \times \bar{A} \quad (4.5-2)$$

Since there are a triply infinite number of closed paths about a point—an infinite number of paths about each of the three orthogonal axes passing through the point P —it may seem that finding the maximal ratio would be a formidable task. However, a perfectly straightforward procedure is taken to resolve this difficulty.

First, a component of the curl in an arbitrary direction, say \hat{u}_1 , is found from the above definition. That is, an arbitrarily selected infinitesimal surface, Δa_1 , is chosen with \hat{u}_1 as its normal. This surface is planar and is bounded by an infinitesimal closed path $d\bar{\ell}$, whose direction is taken in the right-hand sense (that is, with the thumb of the right hand in the direction of \hat{u}_1 , the fingers give the direction of the closed path). It is chosen such that the plane of the path contains the point P at which the curl of \bar{A} is desired. In performing the limit, namely $\lim_{\Delta a \rightarrow 0} (\text{circ}(\bar{A})/\Delta a)$, the vector component of the curl of \bar{A} in the \hat{u}_1 direction is determined:

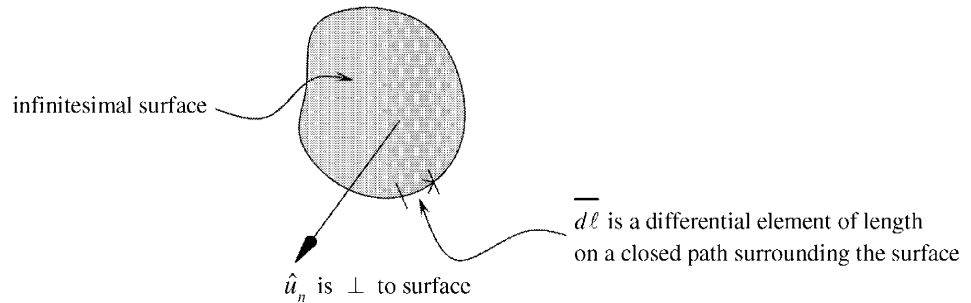


Figure 4.5-1 The geometry associated with the definition of curl.

$$\text{curl } \bar{A}|_1 = \hat{u}_1 \lim_{\Delta a_1 \rightarrow 0} \frac{\oint_{\Delta a_1} \bar{A} \cdot \overline{d\ell}}{\Delta a_1} \quad (4.5-3)$$

A second direction, say \hat{u}_2 , is taken orthogonal to the first but otherwise arbitrary, and the procedure is repeated to obtain the second component. Finally, a third component is taken orthogonal to the first two by the right-hand rule (and, thus, is uniquely determined). We assign its direction as \hat{u}_3 , and repeat the procedure again. Summing the three orthogonal components, the resulting vector is the desired maximal ratio and is the curl of \bar{A} :

$$\begin{aligned} \nabla \times \bar{A} &= \sum_{i=1}^3 \text{curl } A|_i \\ &= \hat{u}_i \sum_{i=1}^3 \lim_{\Delta a_i \rightarrow 0} \frac{\oint_{\Delta a_i} \bar{A} \cdot \overline{d\ell}}{\Delta a_i} \end{aligned} \quad (4.5-4)$$

Note that this expression was determined from the definition without the need for any coordinate system.

4.5.2 The curl as a vorticity vector

In order to give further physical interpretation of the curl operator we need to garner a physical understanding of the *circulation integral*, Eq. (4.5-1) — an intimate part of the definition of the curl. As first presented in Eq. (2.4-22), the circulation of the vector field \bar{A} is

$$\text{circ}(\bar{A}) = \oint \bar{A} \cdot \overline{d\ell} \quad (4.5-5)$$

From [Morse & Feshbach, pp 18ff]¹⁵ *this integral is a measure of the tendency of the field's flow lines to "curl up."* In cases such as magnetic fields or fluid flow fields where the field direction lines either close on themselves or circulate as in a helix, the circulation of the field, $\text{circ}(\vec{A})$, will not be zero. As defined in the discussion following Eq. (2.4-22), such fields are referred to as *rotational*, *solenoidal*, or *nonconservative*. Other terms expressing this circulatory nature of some fields are *paddle-wheeling*^{4,5} (Thomas & Finney, p. 992 and Schwarz, p. 154ff), *swirl*² (O'Neil, p. 972), and *vorticity*¹⁶ (Rogers, p. 275). Each of these terms conjures up the image of circulating or twirling fields.

The *paddle-wheel* concept is perhaps the easiest to understand for the student's initial exposure to curl. Suppose that a small paddle wheel consisting of symmetrical, uniform, planar fins on an axial shaft is placed in a fluid that is flowing. If the flow lines are uniform, that is, having constant direction and strength, the paddle wheel will not rotate no matter what the direction of its axis is. However, if there is a variation in the flow field, either in magnitude or direction or both, there will be orientations of the axis in which the paddle wheel will rotate. The rotational speed of the paddle wheel is a measure of the magnitude of the vector component of the curl. The axis is the direction of the component, where the thumb of the right hand gives the orientation of the direction when the fingers are orientated in the direction of rotation. As the axis is adjusted for maximum rotation, the ultimate curl vector is empirically determined. This postulation may be tested by rotating the axis in each of two orthogonal directions and noting that the paddle wheel does not turn in either of these orientations. Thus, the component of the curl that exhibits maximum circulation where the other two orthogonal components are zero *is* the curl.

Such a *gedanken experiment* (German for "thought experiment") may be tested by the construction of a *curl meter*, which consists of a small paddle wheel metered to display its angular velocity. As with most such instruments, the presence of the probe may affect the field that it measures; however, the instrument can often be oriented to minimize such errors.

The curl operator is a measure of the *circulation density* or *vorticity* of a vector field¹⁵—that is, the circulation per unit cross-sectional area—which is precisely given in the definition of the curl, Eq. (4.5-1). As Morse & Feshbach point out, the limiting process of Eq. (4.5-1) "is more complicated than that used to define the divergence, for the results obtained depend on the orientation of the element of area," another way of pointing out the ultimate task of determining the *maximal* ratio specified by the definition. In their ensuing discussion Morse &

4.5.4 The expansion of the curl in cylindrical coordinates

Substituting r, ϕ, z for q_1, q_2, q_3 and $1, r, 1$ for h_1, h_2, h_3 in Eq. (4.5-12a) we have

$$\nabla \times \bar{A}|_{cyl} = \hat{u}_r \left[\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right] + \hat{u}_\phi \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right] + \hat{u}_z \frac{1}{r} \left[\frac{\partial(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right] \quad (4.5-13a)$$

or alternatively in determinant form, we have

$$\nabla \times \bar{A}|_{cyl} = \begin{vmatrix} \hat{u}_r & \hat{u}_\phi & \hat{u}_z \\ r & & r \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix} \quad (4.5-13b)$$

4.6 Tensorial Resultants of First-Order Vector Differential Operators

To summarize, let us tabulate the resultant quantities from the three first-order vector differential operators developed in the preceding three sections. We will first establish single-character symbols—**D**, **C** and **G**—to denote divergence, curl and gradient, respectively. This ordering is chosen in increasing order of resultant tensor rank. That is, the divergence, curl, and gradient change the rank of the operand—the quantity upon which they operate—by $-1, 0, +1$, respectively. As stated in Section 4.1, a vector differential operator can yield scalar, vector, or tensor fields depending on its properties and depending upon the rank of the operand. Table 4-1 summarizes, encapsulates, and generalizes this statement for the divergence, curl, and gradient of scalars, vectors, dyadics and tensors in general.

Since there can be no quantity with negative rank, the divergence cannot operate on a scalar. Also, by careful inspection of Eq. (4.5-12a), the curl cannot operate on a scalar either. These observations are consistent with the rules for the dot and cross products between vectors. One cannot take a dot or cross product of a vector with a scalar. For the same reason, one cannot take the divergence or

Table 4-1 Resultant tensor rank from three first-order vector differential operators.

Diff. Vector Operator	with a scalar ($n_R = 0$) operand s	with a vector ($n_R = 1$) operand v	with a dyadic ($n_R = 2$) operand d	with a tensor of rank n_R operand ${}_{n_R}T$
D	x	s ($n_R = 0$)	v ($n_R = 1$)	${}_{n_R-1}T$
C	x	v ($n_R = 1$)	d ($n_R = 2$)	${}_{n_R}T$
G	v ($n_R = 1$)	d ($n_R = 2$)	t ($n_R = 3$)	${}_{n_R+1}T$

Key: **x** \Rightarrow nonexistent **d** \Rightarrow dyadic ($n_R = 2$) **D**=divergence
 s \Rightarrow scalar ($n_R = 0$) **t** \Rightarrow triadic ($n_R = 3$) **C**=curl
 v \Rightarrow vector ($n_R = 1$) ${}_{n_R}T$ \Rightarrow tensor of rank n_R **G**=gradient

curl* of a scalar. Therefore, these two operations are noted as “nonexistent” in Table 4-1.

Note that resultant quantities align diagonally in Table 4-1. For example, the operations **Gs**, **Cv**, and **Dd** result in vectors, which line up diagonally. Likewise, **Gv**, **Cd**, and **Dt** (the latter, **D** operating on a triadic, tensor of rank $n_R = 3$) also line up diagonally, each having dyadic resultants.

4.7 Second-Order Vector Differential Operators—Differential Operators of Differential Operators

Thus far we have been dealing with the three classical first-order vector differential operators—divergence, curl, and gradient. In this section, we will cover some of the combinations of these. There are logically nine combinations of these three operators, although some may be nonexistent and some may be zero depending upon the quantity being operated on, which we call the *operand*. Again, in order to list and sort these nine combinations, let us use the same single-character symbols, namely **D**, **C**, and **G**, that we use in building the above table, to denote divergence, curl, and gradient, respectively. This ordering is

* Although the curl of a scalar is considered nonexistent, if such an operation did exist in some sense—a pure abstraction—it would be a scalar, since the curl does not change the rank of the quantity upon which it operates.

Table B-1 The Common Four Orthogonal Coordinate Systems

Coordinate Systems→ & Parameters↓	Generalized Curvilinear	Cartesian	Circular Cylindrical	Spherical
Coordinates and their range of values	q_1 q_2 q_3	$-\infty < x < \infty$ $-\infty < y < \infty$ $-\infty < z < \infty$	$0 \leq r \text{ (or } r_s) < \infty$ $0 \leq \phi \leq 2\pi$ $-\infty < z < \infty$	$0 \leq r \text{ (or } r_s) < \infty$ $0 \leq \theta \leq \pi$ $0 \leq \phi \leq 2\pi$
Transformation to Cartesian coordinates	$x = q_1$ $y = q_2$ $z = q_3$		$x = r \cos \phi$ $y = r \sin \phi$ $z = z$	$x = r \sin \theta \cos \phi$ $y = r \sin \theta \sin \phi$ $z = r \cos \theta$
Orthogonal Unit Vectors	$\hat{u}_1, \hat{u}_2, \hat{u}_3$	$\hat{u}_x, \hat{u}_y, \hat{u}_z$	$\hat{u}_r, \hat{u}_\phi, \hat{u}_z$	$\hat{u}_r, \hat{u}_\theta, \hat{u}_\phi$
Differentials of Coordinates	dq_1, dq_2, dq_3	dx, dy, dz	$dr, d\phi, dz$ or $dr_s, d\phi, dz$	$dr, d\theta, d\phi$ or $dr_s, d\theta, d\phi$
Components of the vector A	A_1, A_2, A_3	A_x, A_y, A_z	A_r, A_ϕ, A_z	A_r, A_θ, A_ϕ
Metric, Lamé Coefficients, or scale factors	$h_1(q_1, q_2, q_3)$ $h_2(q_1, q_2, q_3)$ $h_3(q_1, q_2, q_3)$	1 1 1	1 r 1	1 r $r \sin \theta$
Differential Elements of Length	$dl_1 = h_1 dq_1$ $dl_2 = h_2 dq_2$ $dl_3 = h_3 dq_3$	dx dy dz	dr $r d\phi$ dz	dr $r d\theta$ $r \sin \theta d\phi$
Description of Coordinate Surfaces		$x = x$ $y = y$ $z = z$ three orthogonal planes	$\sqrt{x^2 + y^2} = r$ cylinders; $\tan^{-1} \frac{y}{x} = \phi$ halfplanes from the z axis; $z = z$, planes \perp z axis.	$\sqrt{x^2 + y^2 + z^2} = r$ spheres; $\cos^{-1} \frac{z}{r} = \theta$ z-axis cones $\tan^{-1} \frac{y}{x} = \phi$ halfplanes from the z axis.
Coordinate Surface Graphics		Fig. B-4	Fig. B-5	Fig. B-6

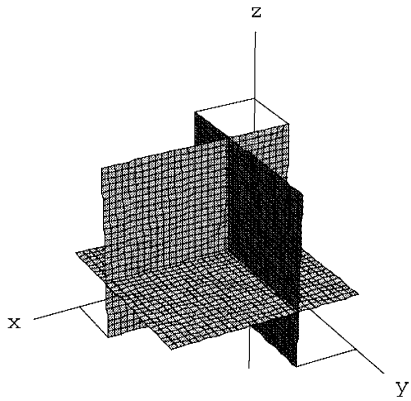


Figure B-4
Cartesian coordinate surfaces

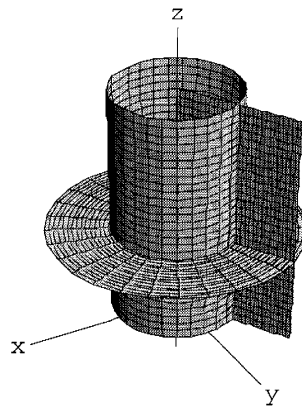


Figure B-5
Cylindrical coordinate surfaces

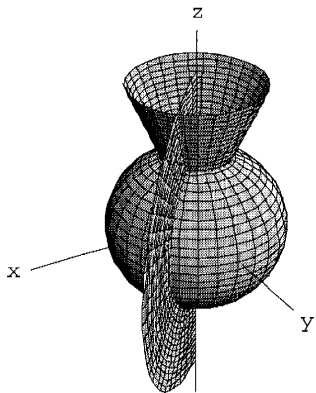
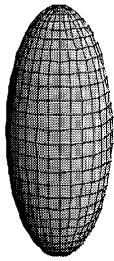


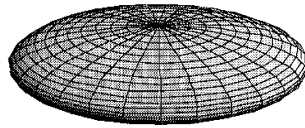
Figure B-6
Spherical coordinate surfaces

Table B-3 Spheroidal Coordinate Systems

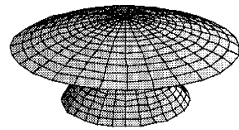
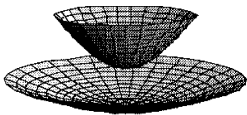
Coordinate Systems→ & Parameters↓	Confocal prolate spheroidal	Confocal oblate spheroidal
Coordinates and their range of values	$\xi \geq 1$ $0 \leq \eta \leq \pi$ $0 \leq \phi \leq 2\pi$	$\xi \geq 1$ $0 \leq \eta \leq \pi$ $0 \leq \phi \leq 2\pi$
Transformation to Cartesian coordinates	$x = c_p \sinh \xi \sin \eta \cos \phi$ $y = c_p \sinh \xi \sin \eta \sin \phi$ $z = c_p \cosh \xi \cos \eta$	$x = c_o \cosh \xi \cos \eta \cos \phi$ $y = c_o \cosh \xi \cos \eta \sin \phi$ $z = c_o \sinh \xi \sin \eta$
Orthogonal Unit Vectors	$\hat{u}_\xi, \hat{u}_\eta, \hat{u}_\phi$	$\hat{u}_\xi, \hat{u}_\eta, \hat{u}_\phi$
Differentials of Coordinates	$d\xi, d\eta, d\phi$	$d\xi, d\eta, d\phi$
Components of the vector A	A_ξ, A_η, A_ϕ	A_ξ, A_η, A_ϕ
Metric Coefficients, Lamé Coefficients, or scale factors	$c_p \sqrt{\sinh^2 \xi - \sin^2 \eta}$ $c_p \sqrt{\sinh^2 \xi - \sin^2 \eta}$ $c_p \sinh \xi \sin \eta$	$c_o \sqrt{\sinh^2 \xi - \sin^2 \eta}$ $c_o \sqrt{\sinh^2 \xi - \sin^2 \eta}$ $c_o \cosh \xi \cos \eta$
Differential Elements of Length	$c_p \sqrt{\sinh^2 \xi - \sin^2 \eta} d\xi$ $c_p \sqrt{\sinh^2 \xi - \sin^2 \eta} d\eta$ $c_p \sinh \xi \sin \eta d\phi$	$c_o \sqrt{\sinh^2 \xi - \sin^2 \eta} d\xi$ $c_o \sqrt{\sinh^2 \xi - \sin^2 \eta} d\eta$ $c_o \cosh \xi \cos \eta d\phi$
Description of Coordinate Surfaces	$\frac{x^2 + y^2}{\sinh^2 \xi} + \frac{z^2}{\cosh^2 \xi} = c_p^2$ prolate ellipsoids; $\frac{x^2 + y^2}{\sin^2 \eta} - \frac{z^2}{\cos^2 \eta} = -c_p^2$ 2-sheet hyperboloids; $\tan^{-1} \frac{y}{x} = \phi$ halfplanes from the z axis.	$\frac{x^2 + y^2}{\sinh^2 \xi} + \frac{z^2}{\cosh^2 \xi} = c_o^2$ oblate ellipsoids; $\frac{x^2 + y^2}{\sin^2 \eta} - \frac{z^2}{\cos^2 \eta} = -c_o^2$ 1-sheet hyperboloids; $\tan^{-1} \frac{y}{x} = \phi$ halfplanes from the z axis.
Coordinate Surface Graphics	Fig. B-10	Fig. B-11



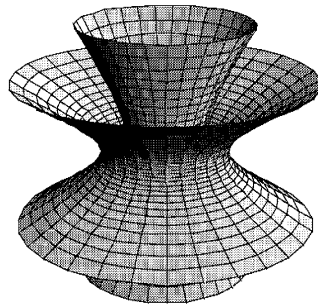
(a) Prolate spheroid



(a) Oblate spheroid



(b) Two-sheet hyperboloids



(b) One-sheet hyperboloids

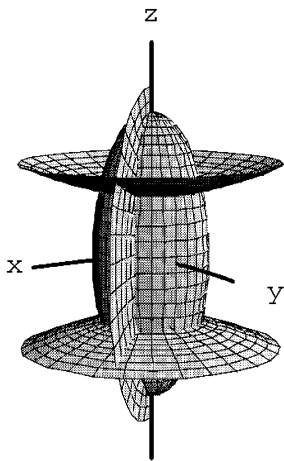


Figure B-10
Prolate spheroidal surfaces

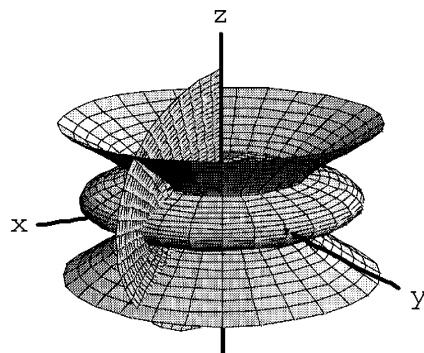


Figure B-11
Oblate spheroidal surfaces