1 Mathematical Preliminaries

We shall go through in this first chapter all of the mathematics needed for reading the rest of this book.

The reader is expected to have taken a one-year course in differential and integral calculus.

1.1 Mean-Value Theorems of Integral Calculus

First mean-value theorem of integral calculus
Let \( f(x) \) be continuous on \( [a, b] \) and \( g(x) > 0 \) (or \( g(x) < 0 \)) in \( [a, b] \).

Then,
\[
\int_a^b f(x)g(x)\,dx = f(x_1) \int_a^b g(x)\,dx,
\]
where \( x_1 \) is in \( [a, b] \).

Proof
We shall prove the case for \( g(x) > 0 \); the case for \( g(x) < 0 \) is entirely analogous.

Since \( f(x) \) is continuous on \( [a, b] \), it is bounded, i.e., there exist \( m \) and \( M \) such that \( m \leq f(x) \leq M \) for all \( x \) in \( [a, b] \).

We further have \( mg(x) \leq f(x)g(x) \leq Mg(x) \) for all \( x \) in \( [a, b] \) since \( g(x) > 0 \) for all \( x \) in \( [a, b] \).

Hence,
\[
m \int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M \int_a^b g(x)\,dx.
\]

\[
m \leq \frac{1}{L_g} \int_a^b f(x)g(x)\,dx \leq M; \quad L_g = \int_a^b g(x)\,dx. \tag{1}
\]

Since \( f(x) \) is continuous on \( [a, b] \), it must evolve continuously between \( m \) and \( M \).

Hence, for any \( y_1 \) satisfying \( m \leq y_1 \leq M \), there exists an \( x_1 \) in \( [a, b] \) such that \( f(x_1) = y_1 \).

Now, apply the above statement to (1).

Let
\[
\frac{1}{L_g} \int_a^b f(x)g(x)\,dx = y_1.
\]
Since $m \leq y_1 \leq M$, there exists $x_1$ in $[a, b]$ such that $f(x_1) = y_1$.

Hence,
\[
\frac{1}{b-a} \int_a^b f(x)g(x)\,dx = f(x_1).
\]

That is,
\[
\int_a^b f(x)g(x)\,dx = f(x_1) \int_a^b g(x)\,dx.
\]

In particular, if $g(x) = 1$, we have
\[
\int_a^b f(x)\,dx = f(x_1) \int_a^b 1\,dx = f(x_1)(b - a).
\]

**Second mean-value theorem of integral calculus**

Let $f(x)$ be monotonically increasing (or decreasing) on $[a, b]$ and $g(x)$ be integrable on $[a, b]$.

Then,
\[
\int_a^b f(x)g(x)\,dx = f(a) \int_a^{x_1} g(x)\,dx + f(b) \int_{x_1}^b g(x)\,dx,
\]

where $x_1$ is in $[a, b]$.

**Proof**

We assume first that $f(x)$ is monotonically increasing, implying that $f'(x) > 0$ on $[a, b]$.

Let
\[
G(x) = \int_a^x g(x)\,dx + c.
\]

Hence, $G$ is differentiable and thus continuous on $[a, b]$.

\[
\begin{align*}
\int_a^b f(x)g(x)\,dx &= \int_a^b f(x)\,dG(x) \\
&= f(x)G(x) \bigg|_a^b - \int_a^b G(x)f'(x)\,dx \\
&= f(b)G(b) - f(a)G(a) - G(x_1)[f(b) - f(a)] \\
&= f(a)[G(x_1) - G(a)] + f(b)[G(b) - G(x_1)]
\end{align*}
\]
\[ f(a) \int_a^{x_1} g(x) \, dx + f(b) \int_{x_1}^b g(x) \, dx. \]

\section*{1.2 The Delta Function}

\subsection*{Definition}
We define the delta function, denoted conventionally as $\delta(x)$, to be the limit of a sequence of functions in the sense that, if

\[ \lim_{n \to \infty} \int_{\alpha}^{b} D_n(x) \, dx = \frac{1}{2}, \quad b > 0 \]

or

\[ \lim_{n \to \infty} \int_{b}^{\alpha} D_n(x) \, dx = \frac{1}{2}, \quad b < 0, \]

then

\[ \lim_{n \to \infty} D_n(x) = \delta(x). \]

\subsection*{Claim}
\[ \lim_{n \to \infty} \int_{c}^{d} D_n(x) \, dx = 0, \]

where $0 < c < d$ or $c < d < 0$.

\subsection*{Proof}
For $0 < c < d$,

\[ \lim_{n \to \infty} \int_{c}^{d} D_n(x) \, dx = \lim_{n \to \infty} \int_{0^+}^{d} D_n(x) \, dx - \lim_{n \to \infty} \int_{0^+}^{c} D_n(x) \, dx = \frac{1}{2} - \frac{1}{2} = 0. \]

The case for $c < d < 0$ can be proved in a similar way.

\subsection*{1.2.1 Representations of the delta function}

1.)
One representation of the delta function is
\[
\frac{\sin(\beta x)}{\pi x}, \quad \beta \rightarrow \infty.
\]

We use \( \beta \) instead of \( n \) as the index of the sequence of functions since it is not restricted to integers.

This is because

\[
\lim_{\beta \to \infty} \int_{0^+}^{b} \frac{\sin(\beta x)}{\pi x} \, dx = \lim_{\beta \to \infty} \frac{1}{\pi} \int_{0^+}^{b} \frac{\sin(x^\wedge)}{x^\wedge} \, dx = \frac{1}{\pi} \int_{0^+}^{\infty} \frac{\sin(x^\wedge)}{x^\wedge} \, dx = \frac{1}{\pi} \frac{\pi}{2} = \frac{1}{2}.
\]

The evaluation of the last integral is detailed in Appendix 1.1.

To show the reader this trend, we plot the representation for \( \beta \) from 1 (blue) to 5 (red) in steps of 1, as shown in Fig. 1.2-1.

2.)

Another representation of the delta function is

\[
\frac{\beta e^{-\beta^2 x^2}}{\sqrt{\pi}}, \quad \beta \rightarrow \infty.
\]

This is because

\[
\lim_{\beta \to \infty} \int_{0^+}^{b} \frac{\beta e^{-\beta^2 x^2}}{\sqrt{\pi}} \, dx = \lim_{\beta \to \infty} \frac{1}{\sqrt{\pi}} \int_{0^+}^{b} e^{-x^2} \, dx = \frac{1}{\sqrt{\pi}} \int_{0^+}^{\infty} e^{-x^2} \, dx = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \frac{1}{2}.
\]

Figure 1.2-1
3.)
Still another representation of the delta function is
\[
\frac{1}{2\pi} \int_{-\beta}^{\beta} e^{ikx} dk, \quad \beta \to \infty.
\]
This is because, by performing the integration
\[
\frac{1}{2\pi} \int_{-\beta}^{\beta} e^{ikx} dk = \frac{1}{2\pi} \frac{e^{i\beta x} - e^{-i\beta x}}{ix} = \frac{\sin(\beta x)}{\pi x},
\]
we can reduce it to the first representation above.

4.)
Our final example of the representation of the delta function is the following sequence of polynomials:
\[
p_n(x) = a_n (1 - x^2)^n, \quad |x| \leq 1; \quad p_n(x) = 0, \quad |x| > 1, \quad n = 1, 2, 3, \ldots,
\]
where \( a_n \) is a normalization factor defined by
\[
a_n \int_{0^+}^{1} (1 - x^2)^n dx = \frac{1}{2}, \quad n = 1, 2, 3, \ldots.
\]
**Proof**
When \( 0 < b < 1 \), consider the following integral
\[
\int_{0^+}^{b} p_n(x) dx = a_n \int_{0^+}^{b} (1 - x^2)^n dx
\]
\[
= a_n \int_{0^+}^{1} (1 - x^2)^n dx - a_n \int_{b}^{1} (1 - x^2)^n dx
\]
\[
= \frac{1}{2} - a_n \int_{b}^{1} (1 - x^2)^n dx.
\]
On one hand,
\[
\frac{1}{a_n} \equiv 2 \int_{0^+}^{1} (1 - x^2)^n dx > 2 \int_{0^+}^{1} (1 - x)^n dx = 2 \left. \frac{-(1-x)^{n+1}}{n+1} \right|_{0^+}^{1} = \frac{2}{n+1},
\]
\[
a_n < \frac{n+1}{2}.
\]
On the other hand,
\[
\int_{b}^{1}(1 - x^2)^n dx < (1 - b^2)^n(1 - b) < (1 - b^2)^n.
\]

Hence,
\[
a_n \int_{b}^{1}(1 - x^2)^n dx < \frac{n + 1}{2}(1 - b^2)^n \rightarrow 0, \quad n \rightarrow \infty.
\]

Then,
\[
\int_{0^+}^{b} p_n(x)dx = \frac{1}{2}
\]

When \( b \geq 1 \),
\[
\int_{0^+}^{b} p_n(x)dx = \int_{0^+}^{1} p_n(x)dx = a_n \int_{0^+}^{1}(1 - x^2)^n dx = \frac{1}{2}
\]
by the definition of \( p_n(x) \) and \( a_n \). Therefore,
\[
\lim_{n \to \infty} \int_{0^+}^{b} a_n (1 - x^2)^n dx = \frac{1}{2}, \quad b > 0,
\]
which means \( p_n(x), \quad n \to \infty \) is indeed a representation of the \( \delta \) function.

1.2.2 Properties of the delta function

1.) Sifting
When \( b > 0 \), if \( f(x) \) is continuous on \((0, b]\) and \( f(x)|_{x \to 0^+} = f(0^+) \), then
\[
\int_{0^+}^{b} f(x)\delta(x)dx = \frac{1}{2}f(0^+).
\]
Similarly, when \( b < 0 \), if \( f(x) \) is continuous on \([b, 0)\) and \( f(x)|_{x \to 0^-} = f(0^-) \), then
\[
\int_{b}^{0^-} f(x)\delta(x)dx = \frac{1}{2}f(0^-).
\]
In particular,
\[
\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = \frac{1}{2} [f(0^-) + f(0^+)],
\]

which is equal to \( f(0) \) if \( f(x) \) is continuous at \( x = 0 \).

Proof

We first prove the case of \( b > 0 \).

Since \( f(x) \) is continuous, there exists a small enough \( b_1 \) such that \( f(x) \) is monotonic on \((0, b_1]\).

We then apply the second mean-value theorem of integral calculus and get

\[
\int_{0^+}^{b_1} f(x) \delta(x) \, dx = \lim_{n \to \infty} \int_{0^+}^{b_1} f(x) D_n(x) \, dx
\]

\[
= \lim_{n \to \infty} f(0^+) \int_{0^+}^{b_1} D_n(x) \, dx + \lim_{n \to \infty} f(b_1) \int_{b_1}^{b_1} D_n(x) \, dx
\]

\[
= f(0^+) \cdot \frac{1}{2} + f(b_1) \cdot 0.
\]

For \( b > 0 \) in general, we write

\[
\int_{0^+}^{b_1} f(x) \delta(x) \, dx = \int_{0^+}^{b_1} f(x) \delta(x) \, dx + \int_{b_1}^{b} f(x) \delta(x) \, dx = \frac{1}{2} f(0^+) + \int_{b_1}^{b} f(x) \delta(x) \, dx.
\]

Next, we divide \([b_1, b]\) into several sub-intervals in which \( f(x) \) is monotonic.

Let one such sub-interval be \([b_{k-1}, b_k]\).

Then,

\[
\int_{b_{k-1}}^{b_k} f(x) \delta(x) \, dx = \lim_{n \to \infty} \int_{b_{k-1}}^{b_k} f(x) D_n(x) \, dx
\]

\[
= \lim_{n \to \infty} f(b_{k-1}) \int_{b_{k-1}}^{b_k} D_n(x) \, dx + \lim_{n \to \infty} f(b_k) \int_{b_k}^{b_k} D_n(x) \, dx
\]

\[
= f(b_{k-1}) \cdot 0 + f(b_k) \cdot 0.
\]

Hence, adding up all such integrals, we have

\[
\int_{0^+}^{b_1} f(x) \delta(x) \, dx = 0.
\]

Therefore, when \( b > 0 \),

\[
\int_{0^+}^{b} f(x) \delta(x) \, dx = \frac{1}{2} f(0^+).
\]
Similarly, when \( b < 0, \)
\[
\int_{b}^{0} f(x)\delta(x)dx = \frac{1}{2} f(0^-).
\]

If \( f(x) \) is continuous at \( x = 0, \)
\[
\int_{-\infty}^{\infty} f(x)\delta(x)dx = \frac{1}{2} [f(0^-) + f(0^+)] = f(0).
\]

\[\blacksquare\]

Letting in the above equation \( f(x) = g(x + c), \) we have
\[
\int_{-\infty}^{\infty} g(x + c)\delta(x)dx = g(c).
\]

By change of variable \( x + c = x^\alpha, \) we obtain
\[
\int_{-\infty}^{\infty} g(x^\alpha)\delta(x^\alpha - c)dx^\alpha = g(c).
\]

The above expression is the most common form for expressing the sifting property of the delta function.

2.)

**Scaling**
\[
\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \frac{1}{|a|} f(0).
\]

**Proof**
If \( a > 0, \)
\[
\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \int_{-\infty}^{\infty} f(x^\alpha/a)\delta(x^\alpha) \frac{dx^\alpha}{a}
\]
\[
= \frac{1}{a} f(0).
\]

If \( a < 0, \)
\[
\int_{-\infty}^{\infty} f(x)\delta(ax)dx = \int_{-\infty}^{\infty} f(x^\alpha/a)\delta(x^\alpha) \frac{dx^\alpha}{a}
\]
\[
= -\frac{1}{a} \int_{-\infty}^{\infty} f(x^\alpha/a)\delta(x^\alpha)dx^\alpha.
\]
\[ = -\frac{1}{a} f(0). \]

3.)

**Functional**

\[ \delta[g(x)] = \sum \frac{1}{|g'(x_i)|} \delta(x - x_i). \]

**Proof**

\( \delta(x) \) is non-trivial only in the neighborhood of \( x = 0 \).

Thus, for \( \delta[g(x)] \), we can only focus on those tiny intervals centered at \( x_i \)'s where \( g(x_i) = 0 \), and on each such interval, approximate \( g(x) \) by a linear function, i.e.,

\( g(x) \approx g(x_i) + g'(x_i)(x - x_i). \)

Hence,

\[ \int_{-\infty}^{\infty} f(x) \delta[g(x)] dx = \sum \int_{-\infty}^{\infty} f(x) \delta[g'(x_i)(x - x_i)] dx. \]

\[ = \sum \frac{1}{|g'(x_i)|} f(x_i). \]

We may then state, equivalently,

\[ \delta[g(x)] = \sum \frac{1}{|g'(x_i)|} \delta(x - x_i). \]

4.)

**Differentiation**

\[ \int_{-\infty}^{\infty} f(x) \delta'(x - c) dx = -f'(c). \]

**Proof**

\[ \int_{-\infty}^{\infty} f(x) \delta'(x - c) dx = \lim_{\Delta \to 0} \int_{-\infty}^{\infty} f(x) \frac{\delta(x + \Delta/2 - c) - \delta(x - \Delta/2 - c)}{\Delta} dx \]

\[ = \lim_{\Delta \to 0} \frac{f(c - \Delta/2) - f(c + \Delta/2)}{\Delta} \]

\[ = -\lim_{\Delta \to 0} \frac{f(c + \Delta/2) - f(c - \Delta/2)}{\Delta} \]

\[ = -f'(c). \]
1.3 Weierstrass’ Approximation Theorem

Weierstrass’ approximation theorem states that any function which is continuous in an interval can be approximated uniformly by polynomials, i.e., $1, x, x^2, \ldots$, in this interval.

Weierstrass’ approximation theorem can be explained by employing the sifting property of the delta function we have just proved.
Assume that $f(x)$ is continuous in $[c, d]$.
Then,
\[
f(x) = \int_{c-}^{d+} f(u) \delta(u - x) \, du
\]
\[
= \lim_{n \to \infty} a_n \int_{c-}^{d+} f(u)[1 - (u - x)^2]^n \, du,
\]
by employing the polynomial representation of the delta function.
Here, $c^- < c < d < d^+$.
Why is the integration domain $[c^-, d^+]$ larger than $[c, d]$?
If we integrate over $[c, d]$, then at the boundary, e.g., at $c$, we only get $f(c)/2$, not $f(c)$.
After performing the integration in the above equation, we obtain a polynomial of order $2n$.
We need to choose a proper $n$ to meet the required error tolerance.

For an explicit proof of Weierstrass’ approximation theorem, see Appendix 1.2.

1.4 Fourier Transform

We define the Fourier transform as
\[
\mathcal{F}[U(x)] \equiv \int_{-\infty}^{\infty} U(x) e^{-ikx} \, dx = \bar{U}(k)
\]
and the inverse Fourier transform as
\[
\mathcal{F}^{-1}[ar{U}(k)] \equiv \int_{-\infty}^{\infty} \bar{U}(k) e^{ikx} \frac{dk}{2\pi}
\]
$U(x)$ and $\bar{U}(k)$ are called Fourier transform pairs.
The functions $U(x)$ and $\bar{U}(k)$ are generally complex; however, the variables $x$ and $k$ are always real unless otherwise stated.
1.4.1 Fourier transform theorems

1.) Fourier integral theorem
\[ \mathcal{F}^{-1}\left[\mathcal{F}[U(x)]\right] = U(x). \]

**Proof**
\[ \mathcal{F}^{-1}\left[\mathcal{F}[U(x)]\right] = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \int_{-\infty}^{\infty} dx_1 e^{-ikx_1} U(x_1) \]
\[ = \int_{-\infty}^{\infty} dx_1 U(x_1) \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ik(x_1-x)} \]
\[ = \int_{-\infty}^{\infty} U(x_1)\delta(x_1-x)dx_1 \]
\[ = U(x). \]

If \( U(x) \) is discontinuous at \( x \), replace \( \delta(x_1-x) \) by \( D_n(x_1-x) \) in the second-to-last equation and let \( n \to \infty \).

We see that the newly obtained \( U(x) \) is the average of \( U(x) \) in the neighborhood of \( x \).

2.) Linearity theorem
\[ \mathcal{F}[a_1 U_1(x) + a_2 U_2(x)] = a_1 \mathcal{F}(k) + a_2 \mathcal{F}(k). \]

3.) Scaling theorem
\[ \mathcal{F}[U(ax)] = \frac{1}{|a|} \bar{U}(k/a). \]

**Proof**
If \( a > 0 \),
\[ \mathcal{F}[U(ax)] = \int_{-\infty}^{\infty} U(ax)e^{-ikx}dx \]
\[ ax = x_1 \]
\[ = \int_{-\infty}^{\infty} U(x_1)e^{-ikx_1}dx_1 \frac{dx_1}{a} \]
\[ = \frac{1}{|a|} \mathcal{F}[U(x_1)] \]
\[
= \frac{1}{a} \int_{-\infty}^{\infty} U(x_1)e^{-\frac{k}{a}x_1}dx_1
\]

\[
= \frac{1}{a} \tilde{U}(k/a).
\]

If \( a < 0 \),

\[
\mathcal{F}[U(ax)] = \int_{-\infty}^{\infty} U(ax)e^{-ikx}dx
\]

\( ax = x_1 \)

\[
\int_{-\infty}^{\infty} U(x_1)e^{-\frac{ik}{a}x_1}dx_1 \quad \frac{1}{a}
\]

\[
= -\frac{1}{a} \int_{-\infty}^{\infty} U(x_1)e^{-\frac{i}{a}x_1}dx_1
\]

\[
= -\frac{1}{a} \tilde{U}(k/a).
\]

\[
\mathcal{F}[U(x - c)] = e^{-ikc} \tilde{U}(k).
\]

**Proof**

\[
\mathcal{F}[U(x - c)] = \int_{-\infty}^{\infty} U(x - c)e^{-ikx}dx
\]

\[
= e^{-ikc} \int_{-\infty}^{\infty} U(x - c)e^{-ik(x - c)}d(x - c)
\]

\( x - c = x_1 \)

\[
= e^{-ikc} \int_{-\infty}^{\infty} U(x_1)e^{-ikx_1}dx_1
\]

\[
= e^{-ikc} \tilde{U}(k).
\]
5.) \textbf{Rayleigh's (Parseval's) theorem}

\[ \int_{-\infty}^{\infty} |U(x)|^2 dx = \int_{-\infty}^{\infty} |\bar{U}(k)|^2 \frac{dk}{2\pi} \]

\textbf{Proof}

\[
\int_{-\infty}^{\infty} |U(x)|^2 dx = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \bar{U}(k) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} e^{-ik_1x}\bar{U}^*(k_1) \\
= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \bar{U}(k) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \bar{U}^*(k_1) \int_{-\infty}^{\infty} dx e^{-i(k_1-k)x} \\
= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \bar{U}(k) \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \bar{U}^*(k_1) 2\pi \delta(k_1 - k) \\
= \int_{-\infty}^{\infty} |\bar{U}(k)|^2 \frac{dk}{2\pi}.
\]

\[
\]

6.) \textbf{Convolution theorem}

The convolution of two functions is defined as

\[ U(x) \otimes V(x) \equiv \int_{-\infty}^{\infty} U(x - x_1)V(x_1) dx_1 \]

\[ x - x_1 = x_2 \]

\[ = \int_{-\infty}^{\infty} V(x - x_2)U(x_2)(-dx_2) \]

\[ = \int_{-\infty}^{\infty} V(x - x_2)U(x_2)dx_2 \]

\[ = V(x) \otimes U(x). \]

Then,

\[ \mathcal{F}\{U(x) \otimes V(x)\} = \int_{-\infty}^{\infty} dx e^{-ikx} \int_{-\infty}^{\infty} dx_1 U(x - x_1)V(x_1) \]
\[ = \int_{-\infty}^{\infty} dx_1 e^{-ikx_1} V(x_1) \int_{-\infty}^{\infty} d(x-x_1) e^{-ik(x-x_1)} U(x-x_1) \]
\[ = \tilde{V}(k) \tilde{U}(k). \]

Besides,

\[ \mathcal{F}[U(x)V(x)] = \int_{-\infty}^{\infty} U(x)V(x)e^{-ikx} dx \]
\[ = \int_{-\infty}^{\infty} dx e^{-ikx} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} e^{ik_1x} \tilde{U}(k_1) \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} e^{ik_2x} \tilde{V}(k_2) \]
\[ = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \tilde{U}(k_1) \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tilde{V}(k_2) \int_{-\infty}^{\infty} dx e^{-i(k-k_1-k_2)x} \]
\[ = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \tilde{U}(k_1) \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tilde{V}(k_2) 2\pi \delta(k-k_1-k_2) \]

either

\[ = \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \tilde{V}(k-k_1) \tilde{U}(k_1) \]
\[ = \tilde{V}(k) \otimes \tilde{U}(k) \]
or

\[ = \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \tilde{U}(k-k_2) \tilde{V}(k_2). \]
\[ = \tilde{U}(k) \otimes \tilde{V}(k). \]

7.)

**Complex conjugate**

\[ \mathcal{F}[U^*(x)] = \int_{-\infty}^{\infty} U^*(x)e^{-ikx} dx \]
\[ = \left[ \int_{-\infty}^{\infty} U(x)e^{ikx} dx \right]^* \]

\[ x = -x_1 \]
\[
\begin{align*}
&= \left[ \int_{-\infty}^{\infty} U(-x_1) e^{-ikx_1} (-dx_1) \right]^* \\
&= \left[ \int_{-\infty}^{\infty} U(-x_1) e^{-ikx_1} dx_1 \right]^* \\
&= [\mathcal{F}[U(-x)]]^* \\
&\neq [\mathcal{F}[U(x)]]^*.
\end{align*}
\]
That is, the operations of the Fourier transform and complex conjugate do not commute unless \( U(-x) = U(x) \), i.e., for functions with inversion symmetry.

8.)

**Autocorrelation theorem**

\[
\mathcal{F}[U(x)\otimes U^*(-x)] = \mathcal{F}[U(x)]\mathcal{F}[U^*(x)]^* = \bar{U}(k)\bar{U}^*(k) = |\bar{U}(k)|^2.
\]

In addition,

\[
\mathcal{F}[|U(x)|^2] = \mathcal{F}[U(x)U^*(x)] = \mathcal{F}[U(x)]\otimes\mathcal{F}[U^*(x)] = \bar{U}(k)\otimes\bar{U}^*(-k).
\]

1.4.2 **Useful Fourier transform pairs**

1.)

**Fourier transform of the rectangle function**

The rectangle function in the real space of width \( W_x \) is defined as

\[
\text{Rect}(x/W_x) = \begin{cases} 
1, & |x| < W_x/2 \\
1/2, & |x| = W_x/2 \\
0, & |x| > W_x/2.
\end{cases}
\]

Finding its Fourier transform is straightforward:

\[
\mathcal{F}[\text{Rect}(x/W_x)] = \int_{-W_x/2}^{W_x/2} e^{-ikx} dx
\]

\[
= e^{-ikx}\bigg|_{x=-W_x/2}^{x=W_x/2} - \left. \frac{e^{-ikx} - e^{ikx}}{-ik} \right|_{x=-W_x/2}^{x=W_x/2}
\]

\[
= \frac{-i2 \sin(kW_x/2)}{-ik}
\]

\[= i \frac{\sin(kW_x/2)}{k/2} = i \frac{2\sin(kW_x/2)}{W_x}.
\]

\[
= i \frac{2\sin(kW_x/2)}{W_x}.
\]
\[\frac{\sin(kW_k/2)}{kW_k/2} = W_k \text{Sinc}(kW_k/2).\]

The rectangular function in the frequency space may be employed more frequently. Similarly, it can be shown that

\[\mathcal{F}^{-1}[\text{Rect}(k/W_k)] = \frac{W_k}{2\pi} \text{Sinc}(W_kx/2).\]

2.)

**Fourier transform of the comb function**

The comb function is defined as

\[\delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x-np),\]

which is a periodic function of period \(p\).

We want to compute its Fourier transform.

\[\tilde{\delta}_p(k) = \int_{-\infty}^{\infty} \delta_p(x)e^{-ikx}dx\]

\[= \sum_{n=-\infty}^{\infty} e^{-i(knp)}\]

\[= \sum_{n=-\infty}^{\infty} e^{i(knp)}\]

\(e^{i(kp)}\) is a periodic function of period \(2\pi/p\).

\(e^{i(k2p)}\) is a periodic function of period \(2\pi/2p\), which is also a periodic function of period \(2\pi/p\).

...  

Therefore, \(\tilde{\delta}_p(k)\) is also a periodic function of period \(2\pi/p\).

\[= \lim_{N \to \infty} \sum_{n=-N}^{N} (e^{ikp})^n \]

\[= \lim_{N \to \infty} \left[\frac{(e^{ikp})^{N+1} - 1}{e^{ikp} - 1}\right] \]

\[= \lim_{N \to \infty} \left[\frac{(e^{ikp})^{N+1} - (e^{ikp})^{-N}}{e^{ikp} - 1}\right] \]

\[= \left(e^{ikp}\right)^{N+1} - \left(e^{ikp}\right)^{-N} = \left[e^{ikp}\right]^{1/2} \left[\left[e^{ikp}\right]^{N+1/2} - \left[e^{ikp}\right]^{-\left(\frac{N+1}{2}\right)}\right]\]
Figure 1.4-1

\[ e^{ikp} - 1 = \left[ e^{ikp} \right]^{1/2} \left[ \left( e^{ikp} \right)^{-1/2} - \left( e^{ikp} \right)^{-1/2} \right] \]

\[ = \lim_{N \to \infty} \frac{\sin[(N + 1/2)kp]}{\sin(kp/2)} \]

\[ \equiv \lim_{N \to \infty} D_N(k). \]

\( D_N(k) \) may diverge at \( k = m(2\pi/p) \) since its denominator equals zero there.

Before going into the mathematical details, we first plot, in Fig. 1.4-1, \( D_N(k) \) versus \( k \) for \( N = 1, 2, 3 \).

(Actually, we plot \( D_N(k) \) versus \( \overline{k} \), defined by \( k = (2\pi/p)\overline{k} \).)

It is seen that as \( N \) increases, the main lobes at \( k = m(2\pi/p) \) become higher (though narrower), whereas the side lobes become lower (if normalized by the main lobe at \( k = 0 \)).

Yes, your guess is correct.

It is an infinite series of delta functions.

We sketch a formal proof below.

**Proof**

First, we consider

\[ \int_{-\pi/p}^{\pi/p} f(k)D_N(k)dk = \int_{-\pi/p}^{\pi/p} f(k) \frac{\sin[(N + 1/2)kp]}{\sin(kp/2)} dk \]

\( (N + 1/2)kp = \nu \)

\[ = \int_{-(N+1/2)\pi}^{(N+1/2)\pi} f\left[ \nu/(N + 1/2)p \right] \frac{\sin \nu}{\sin[v/2(N + 1/2)](N + 1/2)p} d\nu \]

\[ = \frac{2}{p} \int_{-(N+1/2)\pi}^{(N+1/2)\pi} f\left[ \nu/(N + 1/2)p \right] \frac{\sin \nu / \nu}{\sin[v/2(N + 1/2)]/\left[\nu/2(N + 1/2)\right]} d\nu \]
\[
\lim_{N \to \infty} \frac{\sin[v/(N+1/2)]}{v/(N+1/2)} = \lim_{v_1 \to 0} \frac{\sin v_1}{v_1} = 1
\]

\[
\to \frac{2}{p} \int_{-\infty}^{\infty} f(0) \frac{\sin v}{v} dv, \quad N \to \infty
\]

\[
= \frac{4}{p} f(0) \int_{0}^{\infty} \frac{\sin v}{v} dv
\]

\[
\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}
\]

\[
= \frac{2\pi}{p} f(0).
\]

Since \( D_N(k) \) is a periodic function of period \( 2\pi/p \),

\[
\hat{\delta}_p(k) = \lim_{N \to \infty} D_N(k) = \frac{2\pi}{p} \sum_{m=-\infty}^{\infty} \delta[k - m(2\pi/p)].
\]

\[\blacksquare\]

Computing the inverse Fourier transform of the above equation, we obtain

\[
\delta_p(x) = \int_{-\infty}^{\infty} \hat{\delta}_p(k) e^{ikx} dk = \frac{1}{p} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta[k - m(2\pi/p)] e^{ikx} dk
\]

\[
= \frac{1}{p} \sum_{m=-\infty}^{\infty} e^{im \frac{2\pi}{p} x}.
\]

In summary,

\[
\delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np) = \frac{1}{p} \sum_{m=-\infty}^{\infty} e^{im \frac{2\pi}{p} x};
\]

\[
\hat{\delta}_p(k) = \sum_{n=-\infty}^{\infty} e^{i k \cdot np} = \frac{2\pi}{p} \sum_{m=-\infty}^{\infty} \delta[k - m(2\pi/p)].
\]

(1)