Chapter 1
Introduction

1.1 Time-Varying Signals

Time-varying signals are commonly observed in the laboratory as well as many other applied settings. Consider, for example, the voltage level that is present at a specific point in a circuit. In theory, the voltage level can be represented by a real number. More specifically, the voltage level can theoretically be represented by *any real number at any instant in time*, a feature that conforms to the main characteristics of an analog signal. Suppose that an analog voltage level is measured (sampled) using an oscilloscope (o-scope). The o-scope displays a graph of the voltage level, with time depicted in the horizontal direction (t axis) and voltage level in the vertical direction (y axis). Each \((t, y(t))\) point on the o-scope graph represents a measurement of the line voltage at the corresponding time when the measurement was made. The o-scope displays the voltage signal in the *time domain*, i.e., how the voltage varies through time.

As a practical example, suppose a waveform generator is used to create a signal by summing three sinusoids and a noise component. To program the waveform generator, several characteristics of each sinusoid must be entered. A sinusoidal waveform is specified using four critical features: *frequency*, *peak amplitude*, *phase*, and *vertical offset*. A (simplistic) noise component can be specified by declaring its peak amplitude, with the understanding that the noise values are distributed around zero in some manner, e.g., “random (uniform) noise” has a uniform distribution with zero mean, and “band-limited white noise” has a normal distribution with zero mean (see Section 5.2). The noise “amplitude” would represent some characteristic of the
underlying distribution, such as the range or standard deviation of the distribution. As a concrete example, suppose four individual, synthetic waveforms have the characteristics given in Table 1.1. Graphs of the three sinusoidal components are shown separately in Fig. 1.1.

Suppose a waveform generator creates the three individual sinusoids described in Table 1.1 and then adds voltage levels from each sinusoid at every moment in time. Suppose the waveform generator also adds a random (uniform) noise value with a range of ±1 V to the sum of sinusoidal voltages. The resulting sum of all four signal components might look like the curve in Fig. 1.2.

The voltage signal in Fig. 1.2 was constructed by adding four individual components. The specific sinusoids used to construct the signal in Fig. 1.2 are known because the waveform generator was programmed by the user. Consider the following questions:

- With the knowledge of how the signal was built, can the three known sinusoidal components be identified by examining the time-series representation in Fig. 1.2?

<table>
<thead>
<tr>
<th>Component</th>
<th>Frequency</th>
<th>Period (1/Frequency)</th>
<th>Peak Amplitude</th>
<th>Phase Angle</th>
<th>Vertical Offset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sine wave 1</td>
<td>12 Hz</td>
<td>0.0833 s</td>
<td>5.5 V</td>
<td>−141 deg</td>
<td>0 V</td>
</tr>
<tr>
<td>Sine wave 2</td>
<td>50 Hz</td>
<td>0.0200 s</td>
<td>3 V</td>
<td>30 deg</td>
<td>5 V</td>
</tr>
<tr>
<td>Sine wave 3</td>
<td>60 Hz</td>
<td>0.0167 s</td>
<td>1 V</td>
<td>0 deg</td>
<td>0 V</td>
</tr>
<tr>
<td>Random (uniform) noise</td>
<td>–</td>
<td>–</td>
<td>1 V</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 1.1 Specifications for three sinusoidal voltage signals and a random (uniform) noise component that are to be created by a waveform generator.

Figure 1.1 A time-domain depiction of the three sinusoidal voltage signals described in Table 1.1. The individual signals could be created by a waveform generator and added at the output to create a single waveform.
Suppose it is unknown how the signal in Fig. 1.2 was built. Would it be possible to deduce the components that comprise it? In other words, given the signal in Fig. 1.2, is it possible to extract information from the time-domain graph about the sinusoidal signals that were used to construct it?

More specifically, can one determine the frequency, peak amplitude, and phase of each sinusoid that was used to construct the composite signal by examining the waveform in Fig. 1.2? Could one conclude that there is a 60-Hz component with a 1-V peak amplitude?

These questions represent the basic motivation for performing spectral analysis of time-series signals: to explore the frequency content of an arbitrary waveform. An examination of the frequency content of a signal can be accomplished by decomposing the signal into a sum of sinusoidal components. In most cases, viewing a waveform as the composite of individual sinusoids can elucidate important features of the signal that are not intuitively apparent in the time-series depiction. For example, 60-Hz noise is, unfortunately, commonly present in voltage signals due to cross-contamination from power supplies; furthermore, it is often difficult to determine whether 60-Hz noise is present simply by looking at a time-series depiction of the signal. Decomposing a signal into sinusoidal components can help answer the question of whether that noise is present. The same frequency-analysis tools that allow us to look for 60-Hz noise are useful for various
objectives that arise in the analysis of time-series signals, some examples of which are illustrated in Chapter 5.

1.2 The Frequency Domain

The graphs in Figs. 1.1 and 1.2 depict voltage signals in the time domain. That is, the graphs show how the signal levels vary through time. Each point on the graph corresponds to an instant in time, and the height of the graph provides a numerical value that represents the signal level at each instant. Time-domain graphs can provide certain information about the signal that might be helpful for some applications. Viewing a signal in the frequency domain can provide insight into the signal that may not be apparent in the time-domain graph. A mathematical theorem developed by Joseph Fourier (1768–1830) defines a relationship that allows a time-domain signal to be described in an alternative way: as a sum of discrete sinusoids. Fourier’s theorem provides a method that can be used to decompose a time-series voltage signal, such as that shown in Fig. 1.2, into a sum of individual sinusoidal components. The individual sinusoids each have a distinct frequency, so Fourier methods are also sometimes described as a process for determining a power spectrum of the time-series signal. The following sections explore the mathematical details of Fourier’s theorem and how it works.

It is worth examining one of the ways in which Fourier’s theorem is often used to portray the frequency content of time-series signals. Suppose the composite waveform in Fig. 1.2 is sampled at discrete points in time over the domain shown, from 0.0 to 0.5 s; the discrete sampling is depicted in Fig. 1.3. Then, using the sampled time-series data as input, a typical form of the discrete Fourier transform (DFT) output is shown in Fig. 1.4. The graph shows the peak amplitude of sinusoidal components at particular discrete frequencies. That is, each point in the graph of Fig. 1.4 represents the characteristics of an individual sinusoidal curve whose frequency is given by the point’s location along the x axis and whose peak amplitude is given by the point’s location along the y axis. To further understand the information contained in the frequency domain graph, suppose that all of the individual sinusoids depicted by the points in Fig. 1.4 are evaluated at each discrete frequency. The sum of all of the sinusoidal
levels at each discrete frequency will produce the original signal values that are the points on the graph in Fig. 1.3.

The horizontal axis of the graph in Fig. 1.4 shows the frequency, in cycles per second, or hertz (Hz). On the vertical axis is a quantity called the peak-amplitude spectral density, in this case with units of volts/root-Hz. The graph in Fig. 1.4 is referred to as a periodogram estimate of the signal’s frequency content. The interpretation of the

![Figure 1.3](image1.png)

**Figure 1.3** Discrete sampling of the waveform in Fig. 1.2. In theory, the continuous signal depicted in Fig. 1.2 is an analog waveform. Each point in the graph of Fig. 1.3 denotes the instantaneous level of the analog signal of Fig. 1.2 at a particular point in time. The waveform in Fig. 1.3 depicts a series of linear interpolants between adjacent points and does not necessarily represent the true analog signal level at intermediate time values; in practice, the connecting line segments are meant to provide a visual approximation of the signal level, depicting the time evolution of the signal where no measured information would be available.

![Figure 1.4](image2.png)

**Figure 1.4** DFT peak-amplitude spectrum of the discretely sampled composite waveform in Fig. 1.3. The depiction of a peak-amplitude spectrum is referred to as a periodogram estimate of the signal's frequency content. The line segments shown are linear interpolants between adjacent points and do not necessarily represent the true spectral content at intermediate frequency values; in practice, the line is meant to provide a visual approximation to the spectral content, where no measured information would be available.
general information provided in the periodogram can be accomplished by recalling how the original waveform was constructed. Each point in the graph represents a sinusoid of a particular frequency and peak amplitude. Based on Table 1.1, one component of the original waveform is a sinusoid with a frequency of 12 Hz and a peak amplitude of 5.5 V. It should be no surprise that the Fourier transform periodogram displays a prominent peak at 12 Hz, with a spectral-density peak amplitude of 5.5703 V/root-Hz; the spectral peak is present because the time-series signal was constructed using a component with those (approximate) characteristics. There are also peaks in the periodogram at 50 Hz and 60 Hz, corresponding to features of the sinusoids used to construct the original signal. The point at 0 Hz is at a level of approximately 5 V/root-Hz, which corresponds to the average direct-current (DC) offset of the original time-series signal.

A waveform generator could be programmed to create the signal shown in Fig. 1.2. If the output of the waveform generator were to be applied to the input of a waveform analyzer (also called a spectrum analyzer), the analyzer would produce a graph similar to the one shown in Fig. 1.4. The waveform analyzer displays an alternative view of its input signal in the frequency domain. That is, the waveform analyzer shows how the content of the signal varies over a range of frequencies. The two expressions of the signal—in the time domain and in the frequency domain—are equivalent in the sense that the signal information content contained in the two representations is the same. That is to say, there is neither any gain of information nor any loss of information by transforming the time-series signal into the frequency domain using Fourier transform methods. Note that the periodogram representation leaves out phase information for each component sinusoid, an issue that will be taken up in Sections 2.10 and 3.5.

Mathematical procedures exist that begin with the numerical values in one domain and convert them to numerical values in the other domain, which will be illustrated in Chapter 3. Depending on the purpose, it may be easier or more appropriate to work in one domain as opposed to the other. One major benefit of using Fourier methods is that representing a signal as the composite of individual sinusoids can often reveal important attributes of the signal that may
not be apparent in the time-series portrayal of the same data. For example, the 60-Hz signal is readily apparent in the frequency-domain representation of Fig. 1.4, whereas for most people the presence of such a signal Hz component is difficult to discern in the time-series graph of Fig. 1.3.

The example presented in this chapter is meant to be straightforward and only demonstrates a small fraction of the many and varied applications that are based on Fourier transform methods. Several examples are presented in Chapter 5 to showcase some of the many potent results that can be achieved using Fourier transform methods to analyze the frequency content of discretely sampled time-series signals.