2.4 A Better Way

There is a better way to represent this last integral, but we need to back up a bit. What exactly is $\mathbf{F}\cdot\mathbf{n}$? Actually, it's just a dot product, but the integral

$$\int_S \mathbf{F}\cdot\mathbf{n}\, dS \tag{2.31}$$

is called the *flux* of **F**. The word "flux" comes from the Latin word *fluxus*, meaning "flow." For example, suppose you have some water flowing through the end of a tube, as represented in Fig. 2.14(a). If the tube is cut straight, the flow is easy to calculate from the velocity of the water (given by **F**) and the geometry of the tube. If you want to express the flow in terms of the mass of water flowing, you can use the density of the water as a conversion. But what if the tube is not cut straight, as shown in Fig. 2.14(b)? In this case, we need to use some more-complicated geometry—vector geometry—to determine the flux. In fact, the flux is calculated using the last integral in the previous section. So, flux is calculable.

Consider an ideal cubic surface with the sides parallel to the axes (as shown in Fig. 2.15) that surround the point (x, y, z). This cube represents our function **F**, and we want to determine the flux of **F**. Ideally, the flux at any point can be determined by shrinking the cube until it arrives at a single point. We will start by determining the flux for a finite-sized side, then take the limit of the flux as the size of the size goes to zero. If we look at the top surface, which is parallel to the (x, y) plane, it should be obvious

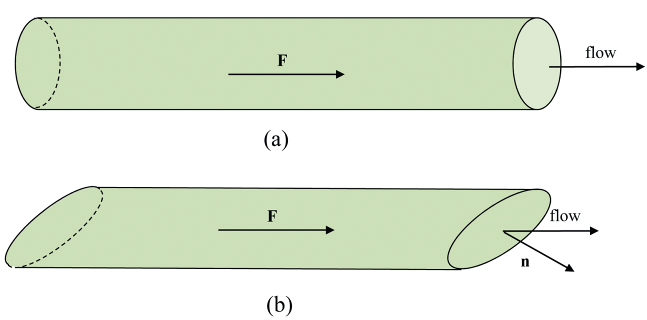

Figure 2.14 Flux is another word for amount of flow. (a) In a tube that is cut straight, the flux can be determined from simple geometry. (b) In a tube cut at an angle, some vector mathematics is needed to determine flux.

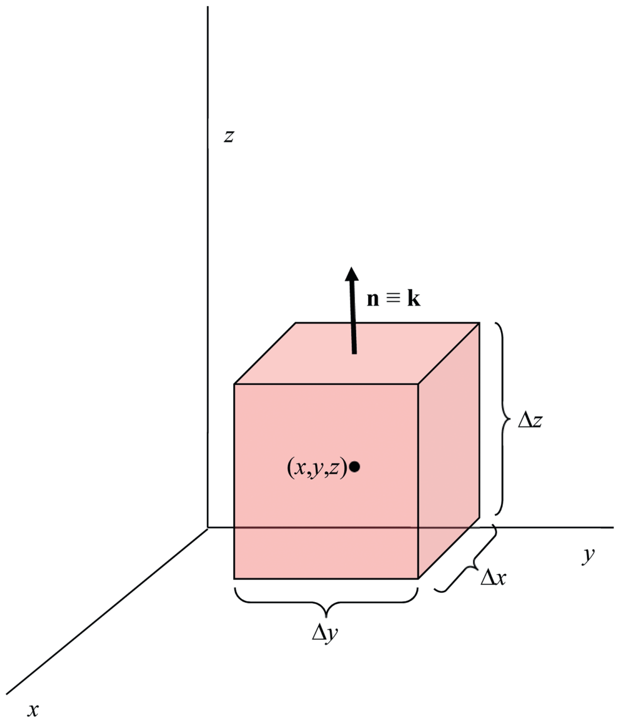

Figure 2.15 What is the surface integral of a cube as the cube gets infinitely small?

that the normal vector is the same as the **k** vector. For this surface by itself, the flux is then

$$\int_S \mathbf{F} \cdot \mathbf{k}\, dS. \tag{2.32}$$

If **F** is a vector function, its dot product with **k** eliminates the **i** and **j** parts (since $\mathbf{i} \cdot \mathbf{k} = \mathbf{j} \cdot \mathbf{k} = 0$; recall that the dot product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$, where $|\mathbf{a}|$ represents the magnitude of vector **a**) and only the z component of **F** remains. Thus, the integral above is simply

$$\int_S F_z\, dS. \tag{2.33}$$

If we assume that the function F_z has some average value on that top surface, then the flux is simply that average value times the area of the surface, which we will propose is equal to $\Delta x \cdot \Delta y$. We need to note, though, that the top surface is not located at z (the center of the cube), but at $z + \Delta z/2$. Thus, we have for the flux at the top surface:

$$\text{top flux} \approx F_z\left(x, y, z + \frac{\Delta z}{2}\right) \cdot \Delta x \Delta y, \tag{2.34}$$

where the symbol $\approx$ means "approximately equal to." It will become "equal to" when the surface area shrinks to zero.

The flux of **F** on the bottom side is exactly the same except for two small changes. First, the normal vector is now – **k**, so there is a negative sign on the expression. Second, the bottom surface is lower than the center point, so the function is evaluated at $z - \Delta z/2$. Thus, we have

$$\text{bottom flux} \approx -F_z\left(x, y, z - \frac{\Delta z}{2}\right) \cdot \Delta x \Delta y. \tag{2.35}$$

The total flux through these two parallel planes is the sum of the two expressions:

$$\text{flux} \approx F_z\left(x, y, z + \frac{\Delta z}{2}\right) \cdot \Delta x \Delta y - F_z\left(x, y, z - \frac{\Delta z}{2}\right) \cdot \Delta x \Delta y. \tag{2.36}$$

We can factor the $\Delta x \Delta y$ out of both expressions. Now, if we multiply this expression by $\Delta z/\Delta z$ (which equals 1), we have

$$\text{flux} \approx \left[F_z\left(x, y, z + \frac{\Delta z}{2}\right) - F_z\left(x, y, z - \frac{\Delta z}{2}\right)\right] \cdot \Delta x \Delta y \frac{\Delta z}{\Delta z}. \tag{2.37}$$

We rearrange as follows:

$$\text{flux} \approx \frac{\left[F_z\left(x, y, z + \frac{\Delta z}{2}\right) - F_z\left(x, y, z - \frac{\Delta z}{2}\right)\right]}{\Delta z} \cdot \Delta x \Delta y \Delta z, \tag{2.38}$$

and recognize that $\Delta x \Delta y \Delta z$ is the change in volume of the cube ΔV:

$$\text{flux} \approx \frac{\left[F_z\left(x, y, z + \frac{\Delta z}{2}\right) - F_z\left(x, y, z - \frac{\Delta z}{2}\right)\right]}{\Delta z} \cdot \Delta V. \tag{2.39}$$

As the cube shrinks, Δz approaches zero. In the limit of infinitesimal change in z, the first term in the product above is simply the definition of the derivative of F_z with respect to z! Of course, it's a partial derivative because **F** depends on all three variables, but we can write the flux more simply as

$$\text{flux} = \frac{\partial F_z}{\partial z} \cdot \Delta V. \tag{2.40}$$

A similar analysis can be performed for the two sets of parallel planes; only the dimension labels will change. We ultimately obtain

$$\begin{aligned}\text{total flux} &= \frac{\partial F_x}{\partial x}\Delta V + \frac{\partial F_y}{\partial y}\Delta V + \frac{\partial F_z}{\partial z}\Delta V \\ &= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta V.\end{aligned} \tag{2.41}$$

(Of course, as Δx, Δy, and Δz go to zero, so does ΔV, but this does not affect our end result.) The expression in the parentheses above is so useful that it is defined as the *divergence* of the vector function **F**:

$$\text{divergence of } \mathbf{F} \equiv \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \text{ (where } \mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}). \tag{2.42}$$

Because divergence of a function is defined at a point, and the flux (two equations above) is defined in terms of a finite volume, we can also define the divergence as the limit as volume goes to zero of the flux density (defined as flux divided by volume):

$$\begin{aligned}\text{divergence of } \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \lim_{\Delta V \to 0} \frac{1}{\Delta V}(\text{total flux}) \\ &= \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_S \mathbf{F} \cdot \mathbf{n}\, dS.\end{aligned} \tag{2.43}$$

There are two abbreviations to indicate the divergence of a vector function. One is to simply use the abbreviation "div" to represent divergence:

$$\text{div}\,\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{2.44}$$

The other way to represent the divergence is with a special function. The function ∇ (called "del") is defined as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{2.45}$$

If we were to take the dot product between ∇ and F, we would obtain the following result:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}\right) \\ &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},\end{aligned} \tag{2.46}$$

which is the divergence! Note that, although we expect to obtain nine terms in the dot product above, cross terms between the unit vectors (such as $\mathbf{i} \cdot \mathbf{k}$ or $\mathbf{k} \cdot \mathbf{j}$) all equal zero and cancel out, while like terms (that is, $\mathbf{j} \cdot \mathbf{j}$) all equal 1 because the angle between a vector and itself is zero and $\cos 0 = 1$. As such, our nine-term expansion collapses to only three nonzero terms. Alternately, one can think of the dot product in terms of its other definition,

$$\mathbf{a} \cdot \mathbf{b} = \Sigma a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3, \tag{2.47}$$

where a_1, a_2, etc., are the scalar magnitudes in the x, y, etc., directions. So, the divergence of a vector function **F** is indicated by

$$\text{divergence of } \mathbf{F} = \nabla \cdot \mathbf{F}. \tag{2.48}$$

What does the divergence of a function mean? First, note that the divergence is a scalar, not a vector, field. No unit vectors remain in the expression for the divergence. This is not to imply that the divergence is a constant; it may in fact be a mathematical expression whose value varies