2.3 Additional Relations

Figure 2.3 identifies additional relations, indicating the locations of the object and image, and the ratio of their heights (magnification) and orientations.

Ray 1 enters the lens parallel to the optical axis. Therefore, the point at which it crosses the axis after exiting the lens is the second focal point F_2. Ray 2 passes through the front focus F_1 and leaves the lens parallel to the optical axis.

The two nodal points are special cardinal points because they lie on the optical axis. A ray aiming at N_1 leaves the lens from N_2 in the same direction as it entered the lens, indicated with ray 3.

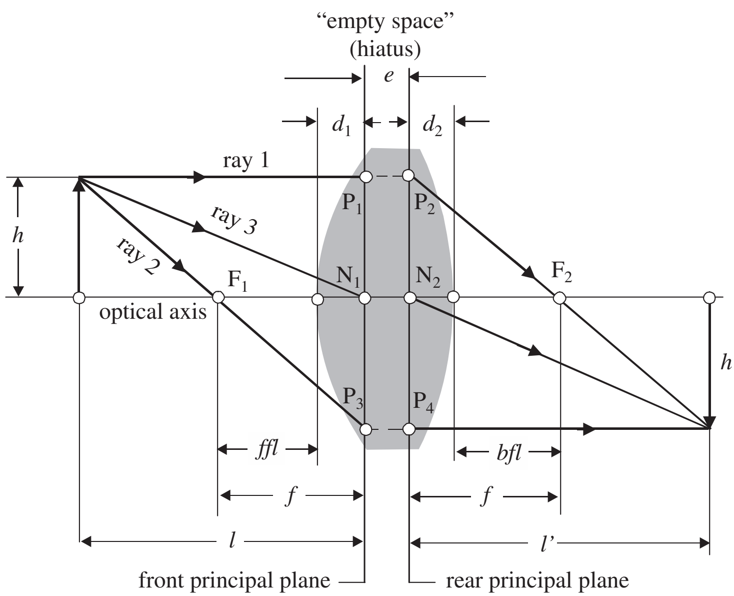

Figure 2.3 Positive lens and its cardinal points. Focal points, F_1 and F_2. Nodal points, N_1 and N_2. Principal points, P_1, P_2, P_3, and P_4.

The principal planes are actually spherical surfaces, but they can be treated as planes in the paraxial region, which is the region close to the optical axis, where the sine and tangent of the ray angles are close to each other, and to the angle expressed in radians.

As a point of reference, $\sin 5\ \text{deg} = 0.08715$, and $\tan 5\ \text{deg} = 0.08748$. Five degrees also equals the 72th part of the 2π full 360-deg circle. Therefore, 5 deg represents $(2\pi/360) \cdot 5 = \pi/36 = 0.08726$ rad. It is a limit of acceptable accuracy in what is termed the paraxial region.

This paraxial treatment is useful even when rays are far from the optical axis because, with the assumption of the trigonometric functions' equality, the equations become linear whereby a simple scaling effect is achieved.

The characteristic feature of the principal planes is that the magnification between them is unity, which means that the rays are transferred at the same height from the front principal plane to the rear principal plane.

In general, a positive lens is used to form an image of an object with a certain magnification.

Sign Convention

- Distances to the left of the front principal plane and heights below the optical axis are negative; distances to the right of the rear principal plane and heights above the optical axis are positive. This indicates that the light is assumed to travel from the left to the right.
- The focal lengths of a positive lens, including the front and back focal lengths, are positive.

> It is extremely important to very carefully apply the agreed-upon sign convention.

The following equations refer to the call-outs in Fig. 2.3:

$$\text{Magnification} \qquad m = \frac{l'}{l} = \frac{h'}{h}. \tag{2.3}$$

$$\text{Back focal length} \qquad bfl = f - \frac{(n-1)tf}{nR_1}. \tag{2.4}$$

$$\text{Front focal length} \qquad ffl = f - \frac{(n-1)tf}{nR_2}. \tag{2.5}$$

The distance from the vertex of the front surface to the front principal plane is

$$d_1 = -\frac{(n-1)tf}{nR_2}, \tag{2.6}$$

and the distance from the vertex of the rear surface to the rear principal plane is

$$d_2 = -\frac{(n-1)tf}{nR_1}. \tag{2.7}$$

To demonstrate what is meant by carefully observing the sign convention, we derive the so-called Gaussian expression* for the location of the image. In Fig. 2.3, it can be seen that the following relations exist:

$$\frac{h}{f} = \frac{(h-h')}{-l'}$$

$$\frac{-h'}{f} = \frac{(h-h')}{-l}$$

***Johann Carl Friedrich Gauss*, a German mathematician, lived from 1777 until 1855 and is well-known for his many contributions in the fields of mathematics and physics.

Rearranging leads to

$$\frac{1}{l'} = \frac{h}{(h-h')} \times \frac{1}{f}, \tag{2.8}$$

$$\frac{1}{l} = \frac{h'}{(h-h')} \times \frac{1}{f}. \tag{2.9}$$

Subtracting Eq. (2.9) from Eq. (2.8) yields

$$\frac{1}{l'} - \frac{1}{l} = \frac{h}{(h-h')} \times \frac{1}{f} - \frac{h'}{(h-h')} \times \frac{1}{f} = \frac{1}{f}\left[\frac{(h-h')}{(h-h')}\right] = \frac{1}{f}.$$

The final Gaussian form is usually presented as

$$\frac{1}{l'} = \frac{1}{l} + \frac{1}{f}. \tag{2.10}$$

To find the image location, one rewrites Eq. (2.10) to read as

$$l' = \frac{lf}{l+f}. \tag{2.11}$$

Exercise 2

Find the image location and height of an object 5 mm high, located 150 mm to the left of the vertex of the lens discussed in exercise 1.

Approach and Solution

Given are $l - d_1 = -150$ mm; $h = 5$ mm. From exercise 1 we know that the focal length $f = 100$ mm, the lens thickness $t = 15$ mm, the index of refraction $n = 1.5$, and the rear radius of the lens $R_2 = -200$ mm.

Using Eq. (2.6), we find the location of the front principal plane

$$d_1 = -\frac{(n-1)tf}{nR_2} = -\frac{(1.5-1)\times 15\times 100}{1.5\times(-200)} = 2.5\ \text{mm}.$$

With that, $l = -150 - d_1 = -150 - 2.5 = -152.5$ mm. Equation (2.11) yields

$$l' = \frac{lf}{l+f} = \frac{-152.5\times 100}{-152.5+100} = \frac{-152{,}500}{-52.5} = 2{,}904.76\ \text{mm}.$$

The magnification is found with Eq. (2.3), i.e., $m = l'/l = 2{,}904.76/(-152.5) \cong -19$.

The image height, also using Eq. (2.3), is $h' = mh = (-19)\times 5 = -95$ mm.

To find the distance of the image from the vertex of the rear surface of the lens, we must subtract d_2 from l'. Using Eqs. (2.7) and (2.11), we obtain

$$\begin{aligned} l' - d_2 = l' - \frac{(n-1)tf}{nR_1} &= 2{,}904.76 \\ &- \frac{(1.5-1)\times 15\times 100}{1.5\cdot 65} = 2{,}897.07\ \text{mm}. \end{aligned}$$

2.4 Negative Lens, Focal Length, and Back Focal Length

A negative lens has a shape as shown in Fig. 2.4. Since its focal length is negative, the locations of the focal points are in reverse order compared to the positive element, as indicated in Fig. 2.4. By choosing the front radius $R_1 = 60$ mm, rear radius $R_2 = 308.35$ mm, thickness $t = 5$ mm, and again