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New analysis of an old instability

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A rigorous mathematical treatment describing the spectral evolution of the modulation instability has applications in optics and fluid dynamics.

Intuitively understanding the meaning of 'noisy' or 'unstable' may be simple, but describing an instability in quantitative terms is considerably more difficult. While the development of advanced computational techniques has led to great progress in the numerical modeling of instabilities, rigorous analytic descriptions are far less common. New results from nonlinear optics, however, have driven significant advances in the description of a very important instability that is relevant to pulse propagation in optical fibers as well as to other nonlinear problems. Such a description opens up new technological possibilities for the design of ultrashort pulse trains (waveforms with a very short wavelength that can be used for data transmission at high bit rates).

The phenomenon in question is referred to as modulation instability (MI). It is a central process of nonlinear physics that has been studied in various forms since the 1960s.[1] A simple way to understand it is to consider a regular but very weak wave perturbation evolving on an otherwise constant background field. When the perturbation frequency falls within a particular range, it is able to draw energy from the background, becoming exponentially amplified and transforming itself into a train of high-contrast peaks in the wave field (see Figure 1). The dramatic effect of high-intensity oscillations emerging from low-power noise is a classic example of the impact that nonlinearity can have on a physical system. This phenomenon has been studied widely in optics from the point of view of ultrafast source development in order to understand performance limitations in optical communications systems.[2]

The vast majority of the previous studies in this area have either used analytical approximations or purely numerical approaches to study the emergence and growth of unstable waves from the background. The issue with developing a more rigorous theory is that the amplification of the initial noise perturbation generates new sidebands. As the number of

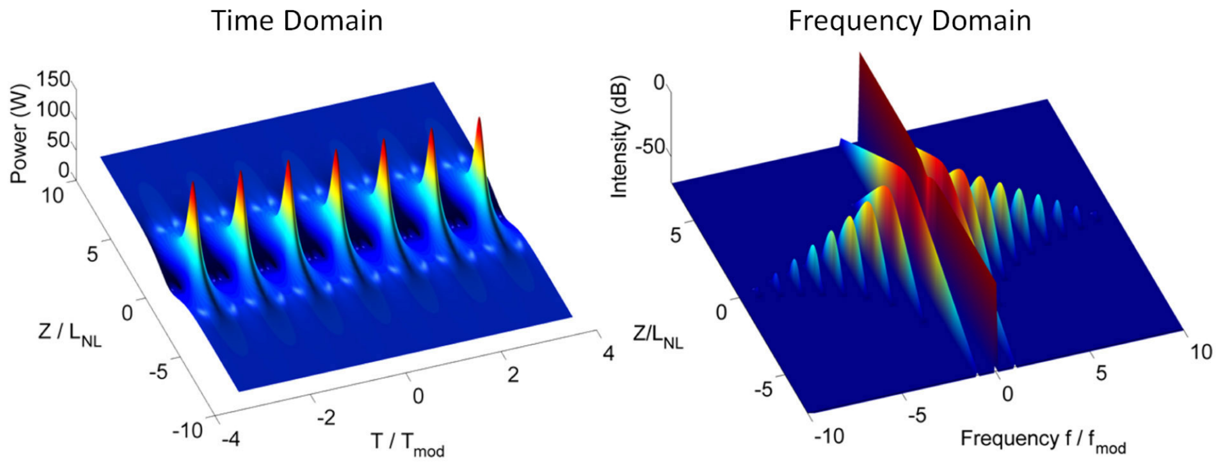
Time Domain
Frequency Domain

Figure 1. *A low-amplitude, periodic perturbation can be amplified to form a train of high-intensity pulses through the modulation instability. The left panel shows how a small temporal modulation on a constant background (not visible on the scale used in the figure) evolves into a train of high power pulses at* $z = 0$ *before decaying. In the frequency domain (right panel), this modulation is associated with multiple harmonic generation. Distance is plotted relative to a characteristic nonlinear length (*L_{NL}*). Time and frequencies are normalized relative to the fundamental modulation period (*T_{mod}*) and frequency (*f_{mod}*) respectively.*

Continued on next page

these harmonics increases with propagation and amplification, standard mathematical techniques using coupled wave equations become intractable.

Surprisingly, however, the literature shows that alternative methods to study unstable waves had already been developed in the 1970s and 1980s. In fact, one particular paper considered the case of nonlinear propagation in optical fibers and derived a simple and exact solution to this problem capable of describing an arbitrary number of generated wavelengths.[3] While the precise mathematical details of this work are somewhat involved, the basic findings can be described simply. As a modulated oscillation propagates in a nonlinear fiber, energy drifts from the central pump wave (the exciting injected wave) to the outer sidebands.

The main result of our analysis has now been to derive specific and relatively simple analytical formulae that completely describe the way in which this transfer occurs. We were able to explain how the energy drift happens for essentially arbitrary injected initial conditions with varying pump power and modulation frequency. By generalizing an approach first described over 20 years ago, we now have available simple and accurate expressions that allow the energy at any particular frequency in the evolving field to be predicted at any given distance of propagation.[4] Our results can be used to study the evolution of the temporal compression (reduction in the temporal width of the injected wave) and harmonic sideband generation that reproduce many previous studies in an approximation-free manner.

Besides having a mathematical interest, our analysis also has consequences in fundamental and applied optics. The development of a spectral theory of MI has already explained why experiments generating white light supercontinuum spectra almost always begin from a phase of a ‘triangular’ spectrum when plotted on a logarithmic (dB) scale.[5] In this case, the pump energy is transferred from a single narrow frequency component or wavelength to generate a broad cascade of new wavelengths spanning the entire visible spectral region. Although seen in numerous experiments, this triangular characteristic was never explicitly investigated. It was only in 2009 that the application of a special case of the MI spectral theory was able to explain this feature.[5] Our new insight has been to see how the energy exchange from the pump to the outer sidebands occurs with reduced efficiency and over shorter interaction distances such that successively generated sideband amplitudes decrease in an exponential fashion. Mathematically, it is this exponential decay that generates the apparent linear triangular shape when plotted on a dB scale.

The development and application of our spectral theory of MI also had a consequence that we had not anticipated. It was recently realized the nonlinear amplification of noise from MI can be linked closely to the appearance of giant freak waves on the ocean that are generated from surface noise.[6] This connection arises because a nonlinearly induced refractive-index perturbation in an optical fiber behaves like a moving fluid and is described mathematically by the same propagation equation that characterizes waves on deep water. The analogy between the appearance of localized structures in optics and the rogue waves on the ocean is both intriguing and attractive, as it opens up possibilities to explore the extreme value dynamics in a convenient optical environment.[7,8]

Our spectral theory describes the dynamics of the developing sideband amplitudes at all distances along the evolution. As a result, the distance at which the highest intensity pulses are generated from the nonlinear amplification can be readily predicted. This allows the precise design of experiments aiming to exploit MI for high repetition rate pulse generation.[2] Some preliminary results along these lines have already been reported.[9]

The ability to study the complex nonlinear evolution of MI for pulse train generation using analytical results has been shown to be beneficial. This is particularly true given that previous approaches to the same problem were based on expensive trial-and-error processes (trying different fiber lengths until it worked) or time-consuming repetitive numerical simulations. Future work in this area will explore the dynamics observed where more complicated initial waveforms are injected at higher power, specifically those that contain two or more dominant initial modulation frequencies. The evolution of such multiple-frequency modulations is something that has been explored mathematically, but very few experimental studies have been carried out. The particular aim here will be to investigate new approaches to generating shorter higher power pulses. We anticipate that our results will have implications for compression dynamics in systems such as high-power amplifiers and fiber lasers.

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Continued on next page

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